QCD

AT THE

LARGE HADRON COLLIDER

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XLIX Herbstschule für Hochenergiephysik

Maria Laach, 11-14 September 2017

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QCD AT THE LHC
I: FACTORIZATION

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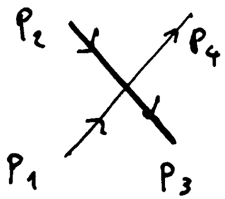
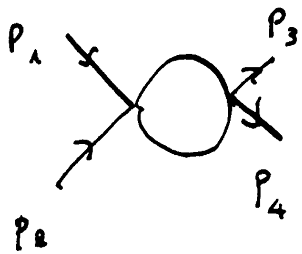
RENORMALIZATION

THE BASIC IDEA

$\mathcal{L}_I = -\frac{g}{24}\phi^4$; $\phi\phi \to \phi\phi$ ELASTIC SCATTERING OF MASSIVE SCALAR FIELDS



$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi}\frac{1}{s}; \qquad s = (p_1 + p_2)^2$$



+ 2 more

$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi}\frac{1}{s}F(s,t):\ \text{DIVERGES!};$$

$$t = (p_1 - p_3)^2,\ u = (p_1 - p_4)^2$$

$$F(s,t) = \lim_{\Lambda\to\infty} 1 + \frac{g}{32\pi}\left(3 + \int_0^1 dx \ln\frac{M^2(s)}{\Lambda^2} + s \to t + s \to u\right); \quad M^2(s) = m^2 - x(1-x)s$$

RENORMALIZATION: EXPRESS A PHYSICAL OBSERVABLE IN TERMS OF OTHER PHYSICAL OBSERVABLES:

WHAT IS THE CHARGE g? DEFINE $g_{\rm phys}$ FROM $\left.\frac{d\sigma}{d\cos\theta}\right|_{s=\mu^2} = \frac{g^2_{\rm phys}}{128\pi}\frac{1}{\mu^2}$:

$$\frac{d\sigma}{d\cos\theta} = \frac{g^2_{\rm phys}}{128\pi}\frac{1}{s}F(s,t); \quad F(s,t) = 1 + \frac{g_{\rm phys}}{32\pi}\left(\int_0^1 \ln\frac{M^2(s)}{M^2(4\mu^2)} + s \to t + s \to u\right)$$

UV SINGULARITY IS UNIVERSAL $\Rightarrow$ REABSORBED IN DEF. OF THE COUPLING

THE RUNNING COUPLING

THE BASIC IDEA

- PHYSICAL RESULTS CANNOT DEPEND ON RENORMALIZATION SCALE μ_R
- HIGH-ENERGY $\Rightarrow \sigma = \sigma\left(g_{\rm phys}(\mu_R), \frac{s}{\mu_R^2}, s\right)$

  $\Rightarrow \mu_R \frac{d}{d\mu_R}\sigma = 0$ RENORMALIZATION GROUP EQUATION
- DEPENDENCE OF $g^2_{\rm phys}(s)$ ON s UNIVERSAL, PERTURBATIVELY COMPUTABLE:
  $\mu_R^2 \frac{d}{d\mu_R^2} g_{\rm phys}(\mu_R) = \beta(g_{\rm phys}(\mu_R))$
- SOLUTION: $\sigma = \sigma\left(g_{\rm phys}(s), s\right)$; DEP. ON s FIXED BY DIMENSIONAL ANALYSIS

ϕ^4 THEORY

- HE TOTAL CROSS-SECTION: $\sigma = \frac{g^2_{\rm phys}(s)}{128\pi}\frac{1}{s}$
- BETA FUNCTION: $\beta(g) = -\beta_0 \frac{1}{2\pi} g^2 + O(g^3)$; $\beta_0 = -\frac{3}{16\pi}$
- RUNNING COUPLING: $g(s) = \frac{g_{\mu_R^2}}{1+\beta_0 \frac{g_{\mu_R^2}}{2\pi} \ln \frac{s}{\mu_R^2}}$ $\Rightarrow$ GROWS W. ENERGY (CHARGE SCREENING)
- μ^R DEP. MUST CANCEL ORDER BY ORDER IN α_s, RESUMMED INTO RUNNING g

QCD & ASYMPTOTIC FREEDOM

- LAGRANGIAN $\mathcal{L} = -\frac{1}{4} G^a_{\mu\mu} G^a_{\mu\nu} + \sum_i^{n_f} \bar{\psi}_i (i \not{D} - m_i) \psi_i$; $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} g_s A^b_\mu A^c_\nu$; $D_\mu = \partial_\mu - i g_s \lambda^a A^a_\mu$; λ^a GENERATORS OF SU(3); $[\lambda^a, \lambda^b] = i f^{abc} \lambda^c$ ACTING ON ψ

- PERTURBATIVE EXPANSION PARAMETER $\alpha_s \equiv \frac{g_s^2}{4\pi}$

- BETA FUNCTION: $\beta(\alpha_s) = -\beta_0 \frac{1}{2\pi} \alpha_2^2 + O(\alpha^3)$; $\beta_0 = \frac{1}{6} (11 C_A - 2 n_f)$; for $SU(N_c)$, $C_A = N_c$

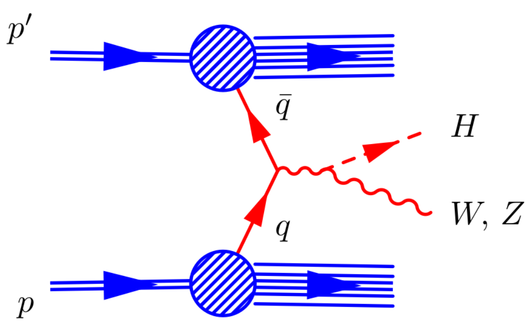
- DO NOT CONFUSE N_c GAUGE GROUP & n_f NUMBER OF FERMIONS (QUARKS)

- RUNNING COUPLING: $\alpha(s) = \frac{\alpha_{\mu^2}}{1+\beta_0 \frac{\alpha_{\mu^2}}{2\pi} \ln \frac{s}{\mu^2}} \Rightarrow$ DECREASES AT HIGH ENERGY: ASYMPTOTIC FREEDOM

PARTON MODEL

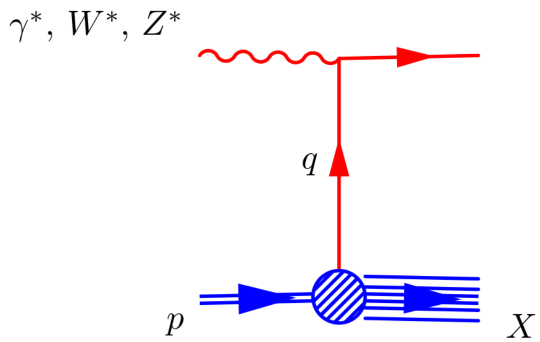
- ASSUME AT HIGH-ENERGY INTERACTIONS DESCRIBED BY FREE-FIELD THEORY
- FUNDAMENTAL SCATTERING HAPPENS AMONG CONSTITUENTS "PARTONS" (QUARKS & GLUONS)
- IF NO RESCATTERING, ALL MOMENTA PARALLEL

HADRON-HADRON



$\Longleftrightarrow$

PHOTON (LEPTON) -HADRON



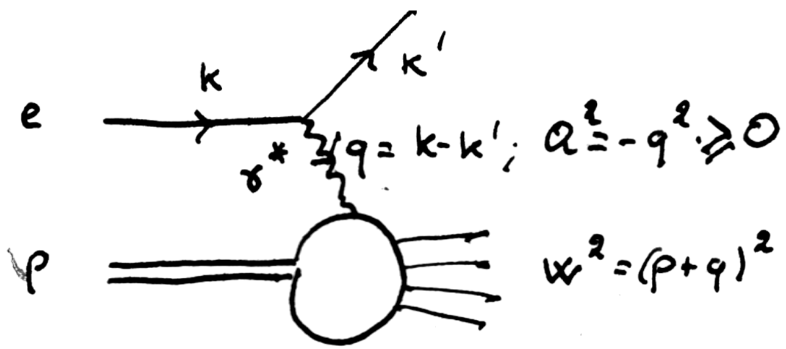
- ONE PARTON PER HADRON: $\hat{p}_1 = x_1 p_1$; $\hat{p}_2 = x_2 p_2$; ($p_i \Rightarrow$ hadrons, $\hat{p}_i \Rightarrow$ partons)
- FOR HADRONIC PROCESSES: $\hat{s} = 2x_1 x_2 p_1 \cdot p_2 = x_1 x_2 s$
  (HIGH-ENERGY, NEGLECTING ALL MASSES)
- FOR LEPTON-HADRON: $2xp \cdot q = x\left[(p+q)^2 - q^2\right] = xW^2 + Q^2$ ($xp \Rightarrow$ parton, q photon)
  $W^2 = (p+q)^2 \equiv s_{\gamma^* p}$, $Q^2 = -q^2 \geq 0$, $x_b \equiv \frac{2p \cdot q}{Q^2}$ (Bjorken x) $\Rightarrow x = x_b$ IF $W^2 = 0$
  (TREE-LEVEL PROCESS)

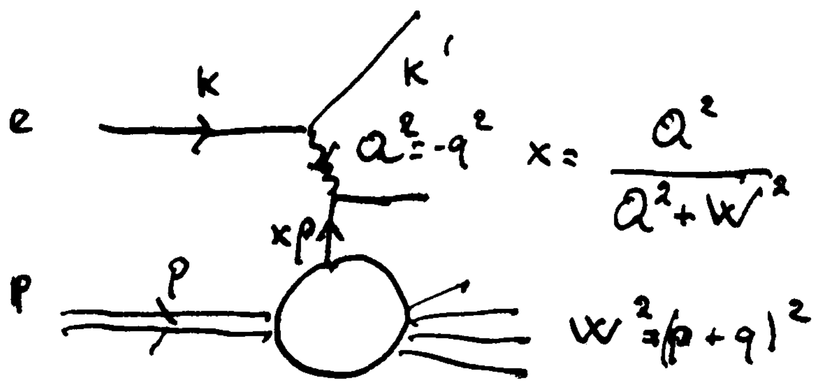
DEEP-INELASTIC $e-p$ SCATTERING

COLLINEAR SINGULARITIES

- LEPTON-PROTON REDUCED TO PHOTON (W, Z)-QUARK; $k' - k = q$
- CROSS-SECTION DEPENDS ON MOMENTA (p, q)
  $\Rightarrow$ SCALE $Q^2 = -q^2$ & DIMENSIONLESS RATIO $x_B = \frac{Q^2}{2p \cdot q}$ (Bjorken variable)
- LEADING-ORDER: $(xp+q)^2 = 0 \Rightarrow x = x_B$; CROSS-SECTION $\sigma \propto \delta(x - x_B)$

LEADING-ORDER



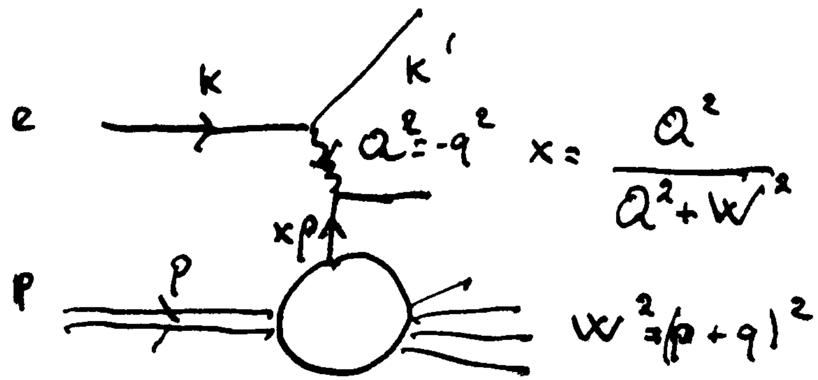


COLLINEAR SINGULARITIES

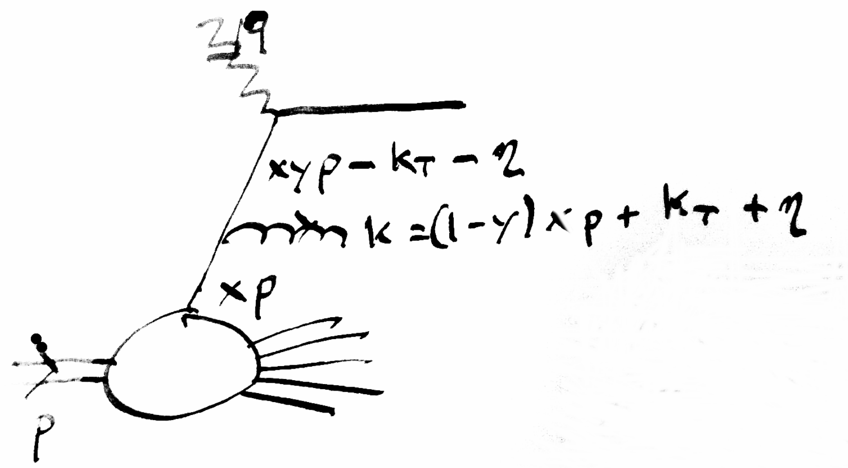
DEEP-INELASTIC $e - p$ SCATTERING

- CROSS-SECTION DEPENDS ON MOMENTA (p, q)
  $\Rightarrow$ SCALE $Q^2 = -q^2$ & DIMENSIONLESS RATIO $x_B = \frac{Q^2}{2p \cdot q}$ (Bjorken variable)
- LEADING-ORDER: $(xp + q)^2 = 0 \Rightarrow x = x_B$; CROSS-SECTION $\sigma \propto \delta(x - x_N)$
- NEXT-TO-LEADING-ORDER: PARAMETRIZE MOMENTUM OF EMITTED QUARK $k = (1 - y)xp + k_T + \eta$ SUCH THAT $k^2 = p^2 = \eta^2 = p \cdot k_t = \eta \cdot k_t = 0$, INTEGRATE INCLUDING PROPAGATOR $\frac{d^3k}{E}\frac{1}{k^2} = dz\frac{d^2k_T}{k_t^2}$ LOG DIVERGENT AS $k_T \to 0$ (COLLINEAR KINEMATICS)

LEADING-ORDER



NEXT-TO-LEADING-ORDER

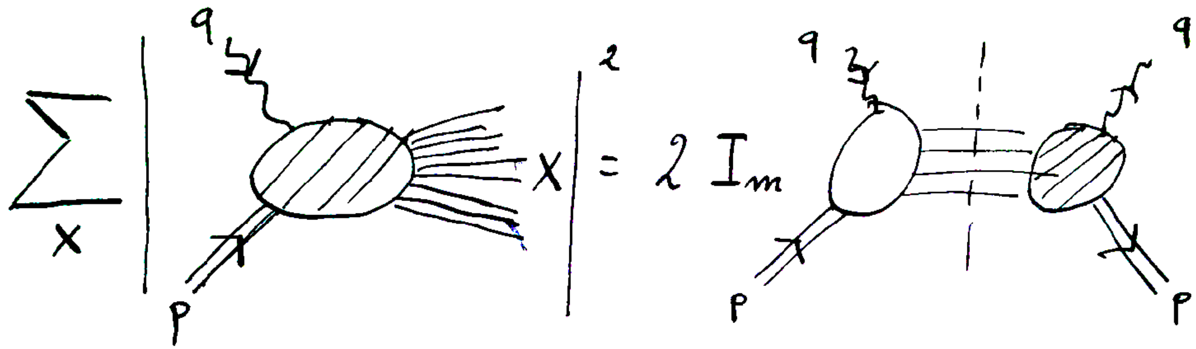


THE WILSON EXPANSION

DEEP-INELASTIC $e-p$ SCATTERING

OPTICAL THEOREM FOR THE INCLUSIVE CROSS SECTION

$$\sigma = \sum_X \left|\langle X|p\gamma^*\rangle\right|^2 = \frac{2}{\pi}\mathrm{Im}\langle p\gamma^*|p\gamma^*\rangle$$



OPERATOR-PRODUCT MATRIX ELEMENT

$$\langle p\gamma^*(q)|p\gamma^*(q)\rangle = i\int \frac{d^4x}{(2\pi)^4} e^{iqx}\langle p|J^\mu(x)J^\nu(0)|p\rangle \equiv W^{\mu\nu}$$

OPERATOR-PRODUCT EXPANSION

$$J^\mu(x)J^\nu(0) = \sum_i C_i(x^2)O_i^{\mu\nu} = \sum_n\sum_k C_{nk}(x^2)O_{nk}^{\mu\nu\alpha_1\ldots\alpha_k}x_{\alpha_1}\ldots x_{\alpha_k}$$

THE WILSON EXPANSION
LEADING-TWIST FACTORIZATION

DIMENSIONAL ANALYSIS: $C_{nk}(x^2) \sim (x^2)^{\frac{d_n}{2}-k-d_j}$; d_n, d_j MASS DIM. OF O_{nk}, J^μ $\Rightarrow$ AS $x^2 \to 0$,

$$J^\mu(x)J^\nu(0) = \sum_{k,\min[d_n-k]} C_{nk}(x^2) O_{nk}^{\mu\nu\alpha_1\ldots\alpha_{k-2}} x_{\alpha_1}\ldots x_{\alpha_k} + O(x^2\Lambda_p^2)$$

Λ_p: CHARACTERISTIC SCALE OF QCD PROTON MATRIX ELEMENTS;
$d_n - k$ TWIST OF THE OPERATOR= dim.-spin

TWIST 2 OPERATORS

$$O_k^{(2,\,q)\,\mu\ldots\alpha_{k-2}} = \bar\psi\gamma^\mu\partial^\nu\partial^{\alpha_1}\ldots\partial^{\alpha_{k-2}}\psi; \quad O_k^{(2,\,g)\,\mu\ldots\alpha_{k-2}} = G_{\mu\alpha}\partial^{\alpha_1}\ldots\partial^{\alpha_{k-2}}G^\nu{}_\alpha$$

- OPERATOR BASIS: SYMMETRIZE AND SUBTRACT TRACE (SPIN EIGENSTATES)
- BEYOND LEADING ORDER $\partial^\mu \to D^\mu$ (GAUGE INVARIANCE)
- COLOR INDICES SUMMED: ONLY COLOR-SINGLET OPERATORS
- ONE FERMION OPERATOR FOR EACH QUARK FLAVOR

OPE MATRIX ELEMENTS

$$i\int\frac{d^4x}{(2\pi)^4}e^{iqx}\langle p|J^\mu(x)J^\nu(0)|p\rangle = \sum_k \langle p|O^{(2,\,q)\,\mu\ldots\alpha_{k-2}}|p\rangle i\int\frac{d^4x}{(2\pi)^4}e^{iqx}C_k^{(2)}(x^2)x_{\alpha_1}\ldots x_{\alpha_k}$$

$$= \sum_k\left(\frac{2p\cdot q}{Q^2}\right)^{k-2}\frac{p^\mu p^\nu}{Q^2}A_K C_k(Q^2)+\ldots$$

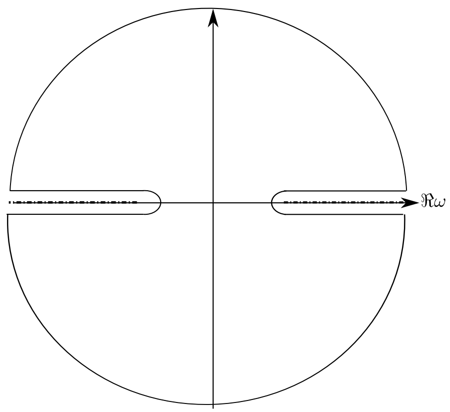
- $\langle p|O^{(2,\,q)\,\mu\nu\alpha_1\ldots\alpha_{k-2}}|p\rangle = A_k 2p^\mu p^\nu p^{\alpha_1}\ldots p^{\alpha_{k-2}}$; A_k reduced mat. el.
- $i\int\frac{d^4x}{(2\pi)^4}e^{iqx}C(x^2)x_\alpha = \frac{2q^\alpha}{Q^2}C(Q^2)$ etc.
- TENSOR STRUCTURE SYMMETRIZED & FIXED BY CURRENT CONSERVATION

FACTORIZATION: FROM MOMENT SPACE TO PHYSICAL SPACE

$$W^{\mu\nu}(x,Q^2) = \sum_k \left(\frac{1}{x}\right)^{k-2} \frac{p^\mu p^\nu}{Q^2} A_K C_k(Q^2) + \ldots$$

- PARAMETRIZE $W\mu\nu$ W. FORM FACTORS W_i:
  $W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \left(p^\mu - q^\mu \frac{p\cdot q}{q^2}\right)\left(p^\nu - q^\nu \frac{p\cdot q}{q^2}\right) \frac{2x}{Q^2} W_2$
- $W_2(x,Q^2) = \sum_k \omega^{k-1} A_k C_k(Q^2)$
  $\Rightarrow A_k C_k(Q^2) = \oint \frac{d\omega}{2\pi i} \omega^{-k} W_2(\omega, Q^2);\ \omega \equiv \frac{1}{x}$
- DEFORM THE INTEGRATION PATH

INTEGRATION PATH



$$\begin{aligned} A_k C_k(Q^2) =& 2\int_1^\infty \frac{d\omega}{2\pi i} \omega^{-k} \left(W_2(x\omega + i\epsilon, Q^2) - W_2(x\omega - i\epsilon, Q^2)\right) \\ =& \frac{2}{\pi}\int_0^1 dx x^{k-2} \mathrm{Im} W_2(x,Q^2) = \int_0^1 dx x^{k-2} F_2(x,Q^2) \end{aligned}$$

$F_i(x,Q^2) = \frac{2}{\pi} \mathrm{Im} W_i(x,Q^2)$:
STRUCTURE FUNCTION (FORM-FACTOR OF XSECT.)

MELLIN TRANSFORMS & CONVOLUTIONS

$h(x) = \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y) \Leftrightarrow h_n = g_n f_n;\quad f_n = \int_0^1 dx x^{n-1} f(x)$, SIMILAR FOR h, g

FACTORIZATION IN PHYSICAL SPACE

$F_2(x,Q^2) = x\int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, Q^2\right) q(y,Q^2);\ \int_0^1 dx x^{n-1} C(x,Q^2) = C_n(Q^2);\ \int_0^1 dx x^{n-1} q(x) = A_n$

COEFFICIENT FUNCTIONS AND PARTON DISTRIBUTIONS

Q: WHAT IS THE MEANING OF $q(x)$, $C(x)$?
A: COMPUTE PERTURBATIVELY!

- TAKE MATRIX ELEMENT IN FREE QUARK STATE: $\langle p|O^{(2,\,q)\,\mu\ldots\alpha_{k-2}}|p\rangle = 2p^{\mu}\ldots p^{\alpha_{k-2}} \Rightarrow$ $A_k = 1$
  EXAMPLE: $k=2 \Rightarrow$ ENERGY-MOMENTUM TENSOR:
  A_2 FRACTION OF THE QUARK ENERGY CARRIED BY ITSELF (!)

- IN PROTON, A_2 IS FRACTION OF PROTON ENERGY CARRIED BY QUARK
  SIMILARLY FOR GLUON OPERATORS, OTHER FLAVORS

- $A_k = 1$ FOR ALL $k \Leftrightarrow q(x) = \delta(1-x)$

COEFFICIENT FUNCTION & PARTONIC CROSS-SECTION

IN A QUARK STATE, $F_2(x,Q^2) = x\int_x^1 \frac{dy}{y} C\left(\frac{x}{y},Q^2\right)\delta(1-y) = xC(x,Q^2) \equiv \hat{\sigma}(x) \Leftrightarrow$

$C(x)$: STRUCTURE FUNCTION (CROSS-SECTION) COMPUTED WITH A QUARK TARGET:
PARTONIC CROSS-SECTION

OPERATOR MATRIX ELEMENTS & PARTON DISTRIBUTIONS

IN A PROTON STATE $\sigma(x) \equiv \frac{F_2(x)}{x} = \int_x^1 \frac{dy}{y}\hat{\sigma}\left(\frac{x}{y},Q^2\right) q(y) = \int_x^1 \frac{dy}{y}\hat{\sigma}\left(\frac{Q^2}{y2p\cdot q},Q^2\right) q(y)$

$q(x)$: RELATIVE WEIGHT OF THE CONTRIBUTION THE CROSS SECTION FROM INITIAL-STATE QUARKS WITH MOMENTUM FRACTION x: PARTON DISTRIBUTION (PDF)
NOTE NOT A PROBABILITY (BEYOND LO), NEGATIVE WEIGHTS ALLOWED!

FACTORIZATION: LEPTON-HADRON VS HADRON HADRON

ONE HADRON IN THE INITIAL STATE

$$\sigma(x) = \int dz \int_x^1 dy \delta(x - yz) q(y) \hat{\sigma}(z) = \int_x^1 \frac{dy}{y} q(y) \hat{\sigma}\left(\frac{x}{y}\right) = \int_x^1 \frac{dy}{y} q\left(\frac{x}{y}\right) \hat{\sigma}(y) = [\sigma \otimes q](x)$$

- $x = x_B$: SCALING VARIABLE FOR HADRONIC PROCESS (MEASURED HADRON KINEMATICS)
- z: SCALING VARIABLE FOR PARTONIC PROCESS (COMPUTED PARTONIC FEYNMAN DIAGRAM)
- y: MOMENTUM FRACTION CARRIED BY INCOMING PARTON
- Q^2 DEP. OF σ, $\hat{\sigma}$ & q OMITTED

TWO HADRONS IN THE INITIAL STATE

$$\sigma(\tau) = \int dz \int_x^1 dx_1 dx_2 \delta(\tau - x_1 x_2 z) q_1(x_1) q_2(x_2) \hat{\sigma}(z) = \int_\tau^1 \frac{dx_1}{x_1} \int_\tau^1 \frac{dx_2}{x_2} dy q_1(x_1) q_2(x_2) \hat{\sigma}\left(\frac{\tau}{x_1 x_2}\right)$$

$$= \int_\tau^1 \frac{dy}{y} \mathcal{L}(y) \hat{\sigma}\left(\frac{\tau}{z}\right) = [\mathcal{L} \otimes \hat{\sigma}](\tau)$$

$$\mathcal{L}(y) \equiv \int_y^1 \frac{dx_1}{x_1} q_1(y) q_2\left(\frac{y}{x_1}\right) = [q_1 \otimes q_2](y)$$

- τ: SCALING VARIABLE FOR HADRONIC PROCESS (MEASURED HADRON KINEMATICS)
- z: SCALING VARIABLE FOR PARTONIC PROCESS (COMPUTED PARTONIC FEYNMAN DIAGRAM)
- x_1, x_2: MOMENTUM FRACTIONS CARRIED BY INCOMING PARTONS
- $\mathcal{L}$: PARTON LUMINOSITY

COLLINEAR SINGULARITIES AND THE RENORMALIZATION GROUP

WHAT HAPPENED TO THE COLLINEAR SINGULARITIES?
WHAT HAPPENS BEYOND LEADING ORDER?

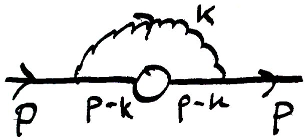
RENORMALIZATION OF OPERATOR MATRIX ELEMENTS

$$A_k 2p^{\mu_1}\ldots p^{\mu_n} = \langle p|O^{(2,\,q)\,\mu_1\cdots\mu_n}|p\rangle = \langle p|\bar\psi\gamma^{\mu}D^{\mu_1}\ldots D^{\mu_n}\psi|p\rangle$$

LEADING ORDER: 

$=\gamma^{\mu_1}\ldots p^{\mu_n} \Rightarrow A_n = 1$

NEXT-TO-LEADING ORDER:



$+4$ more$=\gamma^{\mu}p^{\nu}\ldots p^{\alpha}\mu^{2\epsilon}\Gamma[\epsilon]\frac{\alpha_s}{2\pi}\left[1+4\sum_{j=2}^{n}\frac{1}{j}-\frac{2}{n(n+1)}\right]\Rightarrow$

$$A_n = A_n(\mu^2)$$

RENORMALIZE: $A_n^{\rm ren}(\mu^2) = Z_n^{\rm ren}(\mu_F^2)A_n(\mu^2) = \frac{A_n(\mu^2)}{A_n(\mu_f^2)}$:

$\mu_F^2 \Rightarrow$ SCALE AT WHICH A QUARK IS A SINGLE QUARK: FACTORIZATION SCALE

LOG SCALE DEP.: $\mu_F^2\frac{d}{d\mu_F^2}Z(\mu_F^2) = -\mu_F^2\frac{d}{d\mu_F^2}A_n^{\rm ren}(\mu_F^2) = \mu^2\frac{d}{d\mu}A_n(\mu^2) \equiv \gamma_N$,

ANOMALOUS DIMENSION INDEPENDENT OF μ_F^2 (DIM. ANALYSIS): $\gamma_n = \alpha_s(\mu_R^2)\gamma_N^{(0)} + O(\alpha_s^2)$

RENORMALIZATION GROUP

- $A_k C_k(Q^2) = \int_0^1 dx x^{k-2}F_2(x,Q^2)$ ARE PHYSICAL OBSERVABLES ($\sigma = \frac{F_2}{x}$ IS)
- CANNOT DEPEND ON μ_F: $\mu_F^2\frac{d}{d\mu_F^2}A_kC_k = 0$
- $A_N = A_N(\mu_F^2)$, $C_N = C_N\left(\frac{Q^2}{\mu_R^2},\frac{\mu_F^2}{\mu_R^2},\alpha(\mu_R)\right)$; LET $\mu_F = \mu_R = \mu$, CAN RELAX $\mu_R = k\mu_F$

CALLAN-SYMANZIK (RENORMALIZATION GROUP) EQUATION

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_n(\alpha(\mu^2))\right]C_n\left(\frac{Q^2}{\mu^2},\alpha(\mu^2)\right) = 0$$

ALTARELLI-PARISI EVOLUTION
& COEFFICIENT FUNCTIONS VS PARTON DISTRIBUTIONS
SOLVING THE RGE

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_n(\alpha(\mu^2))\right] C_n \left(\frac{Q^2}{\mu^2}, \alpha(\mu^2)\right) = 0$$

- let $\alpha_s = \alpha(Q^2) \Rightarrow Q^2 \frac{d}{dQ^2} C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right) = \gamma_N \left(\alpha_s(Q^2)\right) C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right)$
- integrate $\Rightarrow C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right) = C\left(1, \alpha_s(Q^2)\right) \exp \int_0^{\ln Q^2} d \ln \mu^2 \gamma_n(\alpha(\mu^2))$
- use α as integration variable $\Rightarrow C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right) = C\left(1, \alpha_s(Q^2)\right) \exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha \frac{\gamma_N(\alpha)}{\beta(\alpha)}$
- recall $C_N \left(\frac{Q^2}{\mu^2}\right) A_N(\mu^2)$ independent of $\mu^2 \Rightarrow$

  $\sigma_n \equiv \int_0^1 dx x^{n-1} \sigma(x, Q^2) = C_n \left(\frac{Q^2}{\mu^2}\right) A_N(\mu^2)$

  $\sigma_n = C\left(1, \alpha_s(Q^2)\right) \left[\exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha \frac{\gamma_N(\alpha)}{\beta(\alpha)}\right] A_n(\mu^2) = \hat{\sigma}_n \left(1, \alpha_s(Q^2)\right) A_n(Q^2)$, WHERE

  $Q^2 \frac{d}{dQ^2} A_n(Q^2) = \gamma_n(\alpha_s(Q^2)) A_N(Q^2) \quad \Leftrightarrow \quad Q^2 \frac{d}{dQ^2} q(,x Q^2) = [P(\alpha_s(Q^2)) \otimes q(Q^2)](x)$

  ALTARELLI-PARISI EQUATION, $\int_0^1 dx x^{n-1} P(x) = \gamma_N$ SPLITTING FUNCTION
- POWER COUNTING: $\hat{\sigma}_n \left(1, \alpha_s(Q^2)\right) = \hat{\sigma}_n^{(0)} + \alpha_s(Q^2) \hat{\sigma}_n^{(1)} + \ldots; \sigma_n^{(0)}$ "PARTON MODEL"

  $\exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha \frac{\gamma_N(\alpha)}{\beta(\alpha)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{\frac{\gamma_n^{(0)}}{\beta_0}} + \cdots = \left(1 + \beta_0 \alpha_s(\mu^2) \ln \frac{Q^2}{\mu^2}\right)^{\frac{\gamma_n^{(0)}}{\beta_0}} + O(\alpha^2 \ln)$

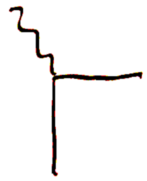
  LEADING LOG

  $= 1 + \alpha_s(\mu^2) \gamma_n^{(0)} \ln \frac{Q^2}{\mu^2} + O(\alpha^2)$ LEADING ORDER

ALTARELLI-PARISI EVOLUTION
& FACTORIZATION OF COLLINEAR SINGULARITIES

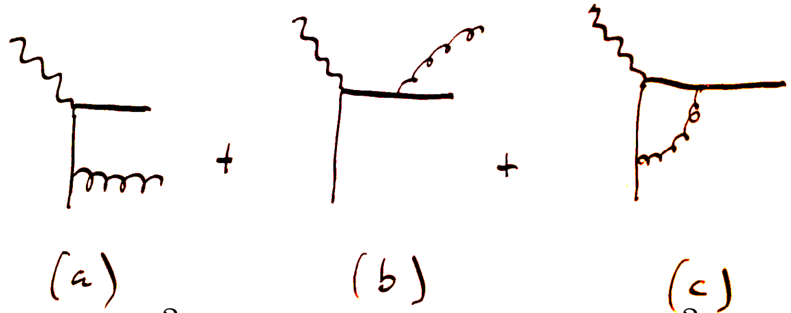
THE HADRONIC CROSS SECTION: $\sigma(Q^2,x) = \hat\sigma(\alpha(Q^2)) \otimes q(Q^2) = \hat\sigma\left(\frac{Q^2}{\mu_F^2}, \alpha(\mu_F^2)\right) \otimes q(\mu_F^2)$

PARTONIC CROSS SECTION: AT LEADING ORDER
$\hat\sigma^{(0)}(x) \propto \delta(1-x) = \Sigma^{(0)}\delta(1-x)$;



$x = x_B \Rightarrow$

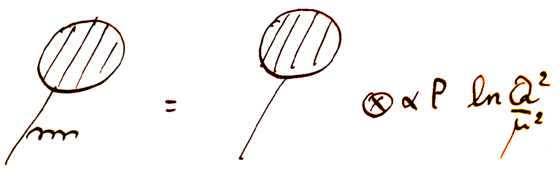
NEXT-TO-LEADING ORDER, LEADING LOG:



$\hat\sigma^{\mathrm{NLO,\,LL}}(\frac{Q^2}{\mu^2}, x) = \hat\sigma^{(0)}(1,x) \otimes \alpha_s(Q^2) P(x) \ln\frac{Q^2}{\mu^2} = \Sigma^{(0)}\alpha_s(Q^2)P(x)\ln\frac{Q^2}{\mu^2}$

WHERE DOES THE LOG COME FROM? DIAGR. (c) $\propto$ LO $\Rightarrow \propto \delta(1-x)$; DIAGR. (b) IS FINITE $\Rightarrow$ NO LOG DEPENDENCE; DIAGR. (a) COLLINEAR DIVERGENCE HAS BEEN FACTORIZED AND REABSORBED IN PARTON DISTRIBUTION;

NEXT-TO-LEADING ORDER (FULL): $\hat\sigma^{\mathrm{NLO}}(\frac{Q^2}{\mu^2}, x) = \alpha_s(Q^2)\hat\sigma^{(1)}(1,x) + \hat\sigma^{\mathrm{NLO,\,LL}}(\frac{Q^2}{\mu^2}, x)$



COLLINEAR FACTORIZATION

- COLLINEAR EMISSION LEADS TO UNIVERSAL PREFACTOR (SPLITTING FUNCTION) CONVOLUTED WITH CROSS-SECTION
- ALTARELLI-PARISI SUMS UP & FACTORIZES EMISSION LADDER
- DIAGRAMMATIC ARGUMENT DOES NOT RELY ON OPE

PARTON KINEMATICS VS. HADRON KINEMATICS

$$\sigma(\tau) = \int_\tau^1 \frac{dy}{y} \mathcal{L}(y) \hat\sigma\left(\frac{\tau}{z}\right); \quad \mathcal{L}(y) \equiv \int_y^1 \frac{dx_1}{x_1} q_1(y) q_2\left(\frac{y}{x_1}\right)$$

$$Q^2 \frac{d}{dQ^2} f_i(,xQ^2) = \sum_j \int_x^1 P_{ij}\left(\alpha_s(Q^2), \frac{x}{y}\right) q_j(Q^2)$$

- q_i QUARKS AND GLUONS; GLUON MIXES WITH SINGLET $\Sigma = \sum_i q_i$
- PARTONIC CHANNEL DEPENDS ON PHYSICAL PROCESS (e.g. $W^+ \Rightarrow u\bar d$ fusion)
- WHICH PARTON MOMENTUM FRACTIONS CONTRIBUTE TO A GIVEN HADRONIC PROCESS ?

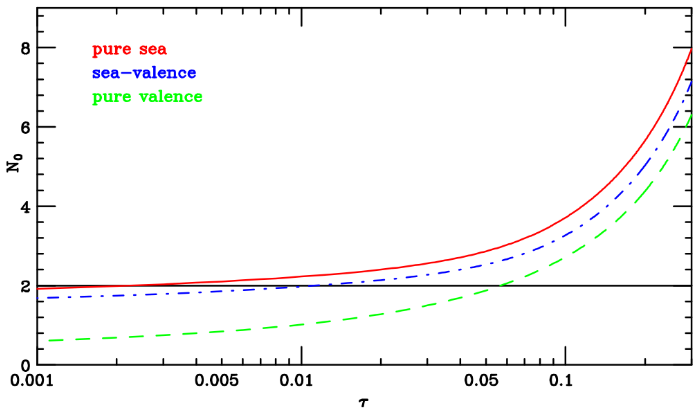
INVERSION OF MELLIN TRANSFORMS

$$f_n = \int_x^1 x^{n-1} f(x) \Leftrightarrow F(x) = \int_{-i\infty}^{+i\infty} x^{-n} f_n$$

integrate to the right of convergence abscissa

- MELLIN INVERSION DOMINATED BY SADDLE POINT
- POSITION OF SADDLE CONTROLLED BY LUMINOSITY
  DEPENDENCE ON x OF $\mathcal{L}$ POWERLIKE, OF $\hat\sigma$ LOGARITHMIC
- PDF PEAKED AT SMALL $x \Rightarrow$ LUMI PEAKS AT SMALL N

SADDLE VS $\tau = Q^2/s$



SUMMARY

- INFINITIES ARISE WHEN EXPRESSING PHYSICAL OBSERVABLES IN TERMS OF ZERO-DISTANCE PHYSICS

  RENORMALIZATION REMOVES THEM EXPRESSING THEM IN TERMS OF OTHER PHYSICAL OBSERVABLES

- FACTORIZATION OF SHORT- AND LONG DISTANCE PHYSICS RELIES ON SCALE SEPARATION

  INTERFERENCE IS POWER-SUPPRESSED

- COLLINEAR SINGULARITIES ARISE WHEN INTEGRATING PERTURBATIVE HARD CROSS-SECTIONS DOWN TO SOFT SCALES

  COLLINEAR FACTORIZATION REABSORBS THEM IN NON-PERTURBATIVE INITIAL CONDITIONS

- FACTORIZATION IS MULTIPLICATIVE IN MELLIN SPACE WHERE HARD CROSS-SECTIONS ARE ANALYTIC FUNCTIONS

  ONE-TO-ONE MAPPING BETWEEN HADRONIC (OBSERVABLE) KINEMATICS AND MELLIN-SPACE VARIABLE IN SADDLE APPROXIMATION