# QCD AT THE LHC

# II: RESUMMATION

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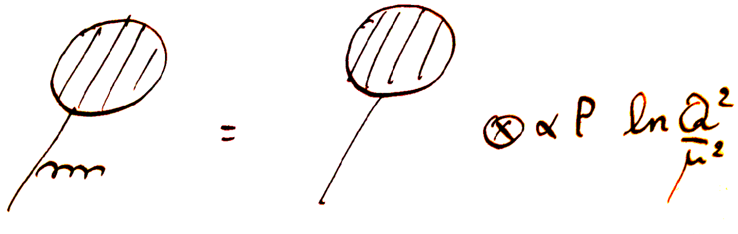
# SUMMARY

## LECTURE II: RESUMMATION

- ALTARELLI-PARISI REVISITED: GLUON EMISSION
- SINGULARITIES AND LOGARITHMS
- RENORMALIZATION GROUP AND EXPONENTIATION OF SOFT LOGARITHMS
- SUDAKOV RESUMMATION AND INFRARED SINGULARITIES
- TRANSVERSE MOMENTUM RESUMMATION AND COLLINEAR SINGULARITIES
- HIGH ENERGY FACTORIZATION AND RESUMMATION
- THE BFKL EQUATION IN COLLINEAR FACTORIZATION

# LARGE LOGS: GLUON EMISSION

FACTORIZED COLLINEAR EMISSION, UP TO NON-LOGARITHMIC TERMS:




$$\sigma^{(1)}(\tau, Q^2) = \int_\tau^1 \frac{dx}{x} \sigma^{(0)}\left(\frac{\tau}{x}, Q^2\right) P(x) \alpha \ln \frac{Q^2}{\mu^2}$$

## THE GLUON SPLITTING FUNCTION

$$P_{gg}(x) = 2C_A \left[ \frac{x}{1-x}_+ + \frac{1-x}{x} + x(1-x) \right] + \beta_0 \delta(1-x)$$

- $x \leftrightarrow 1-x$ SYMMETRY
- THE PARTONIC CROSS-SECTION IS A DISTRIBUTION
- $+$ DISTRIBUTION: $f(x)_+$ ACTS ON $g(x)$ AS $\int_0^1 dx\, f(x)_+ g(x) \equiv \int_0^1 dx\, f(x)[g(x) - g(1)]$

## DIVERGENCES & LOGS

- COLLINEAR: $\int_{\mu^2}^{(s-Q^2)^2/s} \frac{dk_t^2}{k_t^2} \sim \ln\left[\frac{Q^2}{\mu^2} \frac{(1-\tau)^2}{\tau}\right] = \ln \frac{Q^2}{\mu^2} + \ln(1-\tau)^2 + \ln \tau$
- INFRARED (SOFT, LARGE-$x$, SUDAKOV) $\int_\tau^1 dy \frac{1}{1-y}_+ \sim \ln(1-\tau)$
- HIGH-ENERGY (SMALL-$x$, BFKL) $\int_\tau^1 dy \frac{1}{y} \sim \ln(\tau)$

# EIKONAL EMISSION


$$\bar{u}(p) \quad \to \quad \bar{u}(p) ie\gamma^\mu (-i) \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\epsilon}$$

$$= \bar{u}(p) \frac{2p^\mu + (m - \not{p})\gamma^\mu + O(k)}{2p \cdot k + i\epsilon} = \frac{p^\mu}{p \cdot k} \bar{u}(p)$$

- SOFT GLUON EMISSION $\Rightarrow$ UNIVERSAL EIKONAL FACTOR
- COLLINEAR VS. SOFT:
  - SOFT EMISSION IS ALSO COLLINEAR; COLLINEAR EMISSION MAY BE NON-SOFT
  - COLLINEAR EMISSION $\Rightarrow$ UNIVERSAL PARTON-DEPENDENT ALTARELLI-PARISI FACTOR;
  - SOFT GLUON EMISSION $\Rightarrow$ UNIVERSAL EIKONAL FACTOR
- CROSS SECTION FOR SINGLE (DOUBLE. . . ) EMISSION INFRARED DIVERGENT: $\int dk_z \frac{1}{p \cdot x} = \int \frac{dz}{1-z}$
- DIVERGENCE CANCELLED BY VIRTUAL CORRECTIONS: SINGLE EMISSION CANCELLED BY ONE-LOOP, DOUBLE EMISSION CANCELLED BY TWO LOOPS ETC: REAL $\sim \int \frac{dz}{1-z}$; REAL+VIRTUAL $\sim \int \frac{dz}{1-z}_+$
- AFTER CANCELLATION, LEFTOVER SOFT LOGS
- EMISSION FROM A GLUON LINE $\Rightarrow$ SOFT $\leftrightarrow$ HARD SYMMETRY

# SOFT LOGS

## $x$ SPACE VS. $N$ SPACE

$$\int_0^1 dx x^{N-1} \frac{\ln^p(1-x)}{(1-x)_+} = \frac{1}{p+1} \ln^{p+1} \frac{1}{N} + O(\ln^p \frac{1}{N})$$

$$x \to 1 \Leftrightarrow N \to \infty;\ O(1-x) \Leftrightarrow O\left(\frac{1}{N}\right)$$

- EACH EMISSION $\to$ EXTRA FACTOR OF $\frac{1}{(1-x)_+} \Leftrightarrow \ln \frac{1}{N}$
- IN $x$ SPACE, EMISSION CONVOLUTIVE $\to$ IN $N$ SPACE, MULTIPLICATIVE
  MELLIN FACTORIZES PHASE SPACE
- $n$-TUPLE EMISSION $\Rightarrow$ EXTRA FACTOR OF $\ln^n \frac{1}{N} \leftrightarrow \frac{\ln^{n-1}(1-x)}{(1-x)_+}$

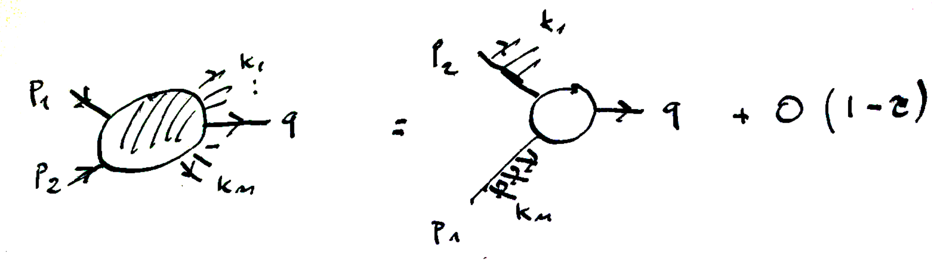
## IRC LOGS

- AT EACH PERTURBATIVE ORDER, TWO SOFT LOGS:
  INFRARED $\int_\tau^1 dy \frac{1}{1-y}_+ \sim \ln(1-\tau)$
  COLLINEAR $\int_{\mu^2}^{(s-Q^2)^2/s} \frac{dk_t^2}{k_t^2} \sim \ln(1-\frac{Q^2}{s})^2 \sim \ln(1-\tau)^2$
- FACTORIZATION SCHEME CHOICE: CAN REDEFINE OPERATOR MATRIX ELEMENT BY ANY FINTE $Z_n(\alpha_s)$: $A_n \to Z_n(\alpha_s) A_n \leftrightarrow$ REDEFINE PDF $q \to Z \otimes q$
- $\overline{\text{MS}}$ (MINIMAL SUBTRACTION) $\Rightarrow$ ONLY $\ln \frac{Q^2}{\mu^2}$ SUBTRACTED FROM $\hat\sigma$ INTO PDF EVOLUTION
- $\overline{\text{MS}}$ PARTONIC XSECT CONTAINS TWO EXTRA SOFT LOGS AT EACH EXTRA PERTURBATIVE ORDER

# THRESHOLD RESUMMATION & RG INVARIANCE I

COLORLESS PRODUCTION (HIGGS, DRELL YAN)

## FACTORIZATION, KINEMATICS & SCALE SEPARATION

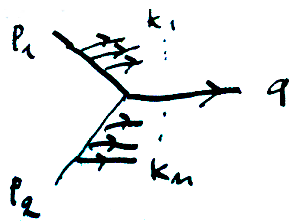



; $q^2 = M^2$; $\tau = \frac{M^2}{s}$; $k^2 \leq \frac{(s^2-M^2)^2}{s}$

- RADIATION FROM INTERNAL LINES POWER-SUPPRESSED IN SOFT LIMIT
- LOOPS:


$= \sigma^0(M^2)H(M^2)$ $H$ "HARD" FUNCTION (LOOP CORRECTIONS TO LEADING-ORDER $\sigma^0$) DIMENSIONLESS, DEPENDS ONLY ON $M^2$

- REAL EMISSION:




$= \sigma^0(M^2)J(M^2(1-\tau)^2)$ "JET" FUNCTION(S) DIMENSIONLESS, DEPENDS ONLY ON $M^2(1-\tau)^2$ IN SOFT LIMIT: PHASE-SPACE!

- $\ln \hat{\sigma}(M^2, \tau) = \ln H(M^2) + \ln J(M^2(1-\tau)^2)$ PARTONIC XSEC FACTORIZES INTO FUNCT. OF $M^2$ (HARD SCALE) & FUNCT. OF $M^2(1-\tau)^2$ (SOFT SCALE)

# THRESHOLD RESUMMATION & RG INVARIANCE II

## COLORLESS PRODUCTION (HIGGS, DRELL YAN)

### RG IMPROVEMENT

- MELLIN TRANSFORM $F(M^2(1-\tau)^2) \Leftrightarrow F\left(\frac{M^2}{N^2}\right)$
- MELLIN-SPACE PARTONIC CROSS-SECTION:
  $\ln\hat\sigma\left(M^2,\mu^2,N,\alpha(\mu^2)\right) = \ln\sigma^0(M^2) + \ln H\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right) + \ln J\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right)$
- $\hat\sigma$ IS NOT RG INVARIANT (NOT PHYSICAL OBSERVABLE); $\hat\sigma\gamma^{\rm phys} = M^2\frac{d}{dM^2}\hat\sigma$ IS RG INVARIANT
  MELLIN SPACE $\hat\sigma$ MULTIPLICATIVELY RENORMALIZED:
  $\sigma^{\rm ren}(N,M^2,\alpha^r(\mu^2)) = Z(N,\alpha^r(\mu^2))\sigma^0(N,M^2,\alpha^0(\mu^2))$
- DEFINE PHYSICAL ANOMALOUS DIMENSION $\gamma^{\rm phys} = M^2\frac{d}{dM^2}\left(\ln H + \ln J\right) = \gamma^c + \gamma^l$
  $\gamma^c = M^2\frac{d}{dM^2}\ln H\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right);\qquad \gamma^l = M^2\frac{d}{dM^2}\ln J\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right)$
- RG INVARIANCE CONDTION $\mu^2\frac{d}{d\mu^2}\gamma^{\rm phys} = 0$ BUT $\gamma^l$, $\gamma^c$ NOT SEPARATELY RGI
- OBSERVATION: $\gamma^l$ DEPENDS ON $M^2/N^2$, $\gamma^c$ DEPENDS ON $M^2$!
  $$\Rightarrow \mu^2\frac{d}{d\mu^2}\gamma^l\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right) = -\mu^2\frac{d}{d\mu^2}\gamma^c\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right) = g(\alpha(\mu^2))$$
  LOOKS LIKE A RG EQUATION!
- SOL: $\gamma^c\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right) = \gamma^c\left(1,\alpha(M^2)\right);\ \gamma^l\left(\frac{M^2/N^2}{M^2},\alpha(M^2)\right) = \int_{M^2}^{M^2/N^2}\frac{d\mu^2}{\mu^2}g\left[\alpha(\mu^2)\right]$

# THE STRUCTURE OF RESUMMED TOTAL CROSS-SECTIONS I

## PARTONIC CROSS SECTION IN THE SOFT LIMIT

$$\hat\sigma\left(N,\frac{M^2}{\mu_F^2},\alpha(\mu^2)\right)=\sigma_0(M^2)H(\alpha(M^2))\exp\int_{\mu_F^2}^{M^2}\int_1^{N^2}\frac{dn}{n}g\left[\alpha(\mu^2/n)\right]=\sigma_0(M^2)C_{\rm res}(N,\alpha_s)$$

RESUMMATION AS RG EVOLUTION DOWN TO SOFT SCALE

## LOG COUNTING

$$C_{\rm res}(N,\alpha_s)=g_0(\alpha_s)\exp\left[\tfrac{1}{\alpha_s}g_1(\alpha_s\ln N)+g_2(\alpha_s\ln N)+\alpha_s g_3(\alpha_s\ln N)+\ldots\right];$$

$$g_0(\alpha_s)=1+\alpha_s g_{0,1}+\alpha_s^2 g_{0,2}+O(\alpha_s^3);\quad g_1(\lambda)=\textstyle\sum_{k=2}^\infty g_{1,k}\lambda^k,\ g_i(\lambda)=\sum_{k=1}^\infty g_{i,k}\lambda^k\ \text{FOR } i\geq 2$$

| LOG APPROX. | XSECT ACCURACY | EXP. ACCURACY: $\alpha_s^n L^k$ | $g_0$ ACCURACY: $\alpha_s^i$ |
|---|---|---|---|
| LL | $k=2n$ | $k=n+1$ | 0 |
| NLL | $2n-2\leq k\leq 2n$ | $k=n$ | 1 |
| NNLL | $2n-4\leq k\leq 2n$ | $k=n-1$ | 2 |

NOTE ACCURACY OF $g_0$ ONE ORDER HIGHER THAN CORRESP. LOG ACCURACY $\Rightarrow$ INCREASES LOG ACCURACY OF $\hat\sigma$ BY ONE ORDER

# THE STRUCTURE OF RESUMMED TOTAL CROSS-SECTIONS II

## RG INVARIANT EXPRESSION

$$C_{\rm res}(N,\alpha_s) = \bar g_0(\alpha_s)\exp\left[\int_1^{N^2}\frac{dn}{n}\int_{M^2 n}^{M^2}\frac{d\mu^2}{\mu^2}\, g\left[\alpha(\mu^2/n)\right]\right]$$
$$= \hat g_0(\alpha_s)\exp\left[2\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\int_{M^2}^{M^2(1-x)^2}\frac{d\mu^2}{\mu^2}\,\hat g\left[\alpha(\mu^2)\right]\right]$$

- TO BE USED WITH PDFS EVALUATED AT $\mu_f^2 = M^2$
- $\hat g$ DETERMINED ORDER BY ORER BY $g$
  $\Rightarrow$ "SOFT SPLITTING FUNCTION VS. "SOFT ANOMALOUS DIMENSION"
- $\bar g_0$, $\hat g_0$ INCLUDE MISSING "CONSTANT" TERMS

## CUSP ANOMALOUS DIMENSION

$$C_{\rm res}(N,\alpha_s) = \hat g_0(\alpha_s)\exp\left[2\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\int_{M^2}^{(1-z)^2M^2}\frac{dq^2}{q^2}A_g^{\rm th}\left(\alpha_s\left(q^2\right)\right)+D_g^{\rm th}\left(\alpha_s\left((1-z)^2M^2\right)\right)\right]$$

- $A$ AND $D$ POWER SERIES IN $\alpha_s$
- LEADING LOG $\Rightarrow$ LEADING ORDER $A \leftrightarrow$ COEFFICIENT OF $\frac{1}{1-x_+}$ IN SPLITTING FUNCTION
- $O(\alpha_s^n)$ CONTRIBUTION TO $A$ DEFINED AS $O(\alpha_s^n)$ COEFFICIENT OF $\frac{1}{1-x_+}$ IN SPLITTING FUNCTION:
  CUSP ANOMALOUS DIMENSION
- $D$ STARTS AT NNLO, DUE TO LARGE-ANGLE GLUON EMISSION
- IF FINAL STATE CAN RADIATE (E.G. DIS), FURTHER $D$-LIKE "$B$" TERM DUE TO FINAL-STATE COLLINEAR RADIATION

# APPLIED THRESHOLD RESUMMATION

- RESUMMATION REQUIRED IF $\alpha \ln \frac{1}{N} \sim 1 \Leftrightarrow$ PROVIDES INFO ON ALL-ORDER $\hat{\sigma}$
- ROUTINELY USED TO IMPROVE ACCURACY OF FIXED-ORDER BY MATCHING TO IT
- PROCESS-DEPENDENT RESUMMATION COEFFICIENTS CAN BE DETERMINED FOR ANY ELECTRO- OR HADRO-PRODUCTION PROCESS
- CAN BE PERFORMED BOTH FOR TOTAL CROSS-SECTIONS RAPIDITY DISTRIBUTIONS, TRANSVERSE-MOMENTUM DISTRIBUTIONS
- $g$ COEFFICIENTS DETERMINED COMPARING TO FIXED ORDER
  $\Rightarrow$ FINITE-ORDER FIXES RESUMMATION AT CORRESPONDING LOGARITHMIC ORDER:
  $$\mathrm{N}^k\mathrm{LO} \Rightarrow \mathrm{N}^k\mathrm{LL}$$
- $A$ DETERMINED FROM SOFT LIMIT OF $P_{qq}$ OR $P_{gg}$
- $B$, $D$ COEFFICIENTS DETERMINED FROM SOFT LIMIT OF HARD CROSS-SECTIONS
- NNNLL AVAILABLE FOR PROCESSES WITH TWO PARTONS AT BORN LEVEL: DRELL-YAN, HIGGS; NNLL AVAILABLE FOR HEAVY QUARKS; GRADUALLY EXTENDED TO PROCESSES WITH THREE OR MORE PARTONS AT BORN LEVEL: PROMPT PHOTONS, $W$, $Z$ AND HIGGS AT HIGH $p_t$, JETS, ONE-PARTICLE INCLUSIVE

# TRANSVERSE MOMENTUM RESUMMATION

## THE $p_t$ DISTRIBUTION

$$P(p_1) + P(p_2) \to H(p) + X$$

$$p = \alpha p_1 + \beta p_2 + p_t; \quad p_t \cdot p_1 = p_t \cdot p_2 = 0$$

### FACTORIZATION

$$Q^2 = \left(\sqrt{M^2 + p_t^2} + p_t\right)^2, \; \tau' = \frac{Q^2}{s}$$

$$\frac{d\sigma}{dp_t^2}\left(\tau', p_t, M^2\right) = \tau' \sum_{ij} \int_{\tau'}^{1} \frac{dx}{x} \mathcal{L}_{ij}\left(\frac{\tau'}{x}, \mu_f^2\right) \frac{1}{x} \frac{d\hat{\sigma}_{ij}}{dp_t^2}\left(x, p_t, \alpha_s, \mu_f^2\right)$$

### THE LEADING-ORDER PARTONIC CROSS-SECTION

$$\frac{d\sigma}{dp_t^2} = \alpha_s \left[\left[\frac{\ln(p_T^2/M^2)}{p_t^2/M^2}\right]_+ P_1 + Q_1(p_t^2/M^2)\right] + \delta(p_t^2/M^2) D_1$$

$p_1$, $D_1$ NUM. COEFFICIENTS; $Q_1$ FUNCTION

- TRANSVERSE MOMENTUM DEPENDENCE REQUIRES AT LEAST ONE EMISSION $\Rightarrow$ STARTS AT $O(\alpha_s)$
- SINGULAR BEHAVIOUR AS $p_t \to 0 \Rightarrow$ COLLINEAR LOG
- CANCELLATION OF IR SING.: $\Rightarrow$ $p_t$ PLUS DISTRIBUTION SAME ORIGIN AS $\frac{1}{1-x}{}_+$
  $\int_0^1 d\xi \, g\,(\xi)\, f\,(\xi)_+ = \int_0^1 d\xi \, \left[g\,(\xi) - g\,(0)\right] f\,(\xi); \quad \xi = p_t^2/M^2$
- RESUMMATION REQUIRED IN ORDER TO OBTAIN RELIABLE PREDICTIONS FOR SMALL $p_t$, $\xi_p \sim 1$

# THE STRUCTURE OF TRANSVERSE MOMENTUM RESUMMATION

PHASE-SPACE FACTORIZATION

LONGITUDINAL $\leftrightarrow$ MELLIN; TRANSVERSE $\leftrightarrow$ FOURIER

$$\frac{d\hat{\sigma}}{dp_t^2}(\alpha_s, p_t^2) = \frac{M^2}{2\pi} \int d^2b\, e^{-i\vec{p}_t\cdot\vec{b}}\, \Sigma(\alpha_s, b^2) = \int_0^{+\infty} db\, b\, J_0(bq_T)\, \Sigma(\alpha_s, b^2)$$

## RESUMMATION

PDFS AT SCALE $M^2$

$$\frac{d\hat{\sigma}_{ij}}{dp_t^2}\left(N, p_t, \alpha_s\left(M^2\right), M^2\right) = \sigma_0 \int_0^\infty db\, \frac{b}{2}\, J_0\left(bp_t\right) H_{ij}\left(N, \alpha_s\left(M^2\right)\right) S(M, N, b)$$

- $ij$ PARTONIC SUBCHANNEL
- RESUMMATION $\Rightarrow$ SUDAKOV EXPONENT

$$S(M, b) = \exp\left[-\int_{\frac{1}{b^2}}^{M^2} \frac{dq^2}{q^2}\left[A^{pt}\left(\alpha_s\left(q^2\right)\right)\ln\frac{M^2}{q^2} + B^{pt}\left(\alpha_s\left(q^2\right), N\right)\right]\right]$$

- LEADING LOG $\rightarrow$ LEADING ORDER $A$; DEFINES $A$-$B$ SEPARATION

  NOTE BEYOND NNLL $A$ DIFFERS FROM CUSP ANOMALOUS DIMENSION

## HARD FUNCTION

$$H_{ij}(\alpha_s) = \left[C_i(N, \mathbf{b})C_i(N, \mathbf{b}) + G_i(N, \mathbf{b})G_j(N, \mathbf{b})\right]$$

$C$, $G$ UNIVERSAL (DEP. ON PARTON)

# THE STRUCTURE OF TRANSVERSE MOMENTUM RESUMMATION

## PHASE-SPACE FACTORIZATION

LONGITUDINAL ↔ MELLIN; TRANSVERSE ↔ FOURIER

$$\frac{d\hat{\sigma}}{dp_t^2}(\alpha_s, p_t^2) = \frac{M^2}{2\pi}\int d^2b\, e^{-i\vec{p}_t\cdot\vec{b}}\,\Sigma(\alpha_s, b^2) = \int_0^{+\infty} db\, b\, J_0(bq_T)\,\Sigma(\alpha_s, b^2)$$

## RESUMMATION

PDFS AT SCALE $1/b^2$

$$\frac{d\hat{\sigma}_{ij}}{d\xi_p}\left(N, \xi_p, \alpha_s\left(M^2\right), M^2\right) = \sigma_0 \int_0^\infty db\, \frac{b}{2}\, J_0\left(bp_t\right) H_{ij}\left(N, \alpha_s\left(M^2\right)\right)\bar{S}(M, b)$$

- RESUMMATION ⇒ SUDAKOV EXPONENT

$$\bar{S}(M, b) = \exp\left[-\int_{\frac{b_0^2}{b^2}}^{M^2}\frac{dq^2}{q^2}\left[\bar{A}^{p_t}\left(\alpha_s\left(q^2\right)\right)\ln\frac{M^2}{q^2} + \bar{B}^{p_t}\left(\alpha_s\left(q^2\right)\right)\right]\right]$$

- $B$ NOW $N$-INDEPENDENT
- $\bar{A}$ AND $\bar{B}$ DETERMINED FROM $A$, $B$, $\beta$ FUNCTION & ANOMALOUS DIMENSIONS

## HARD FUNCTION

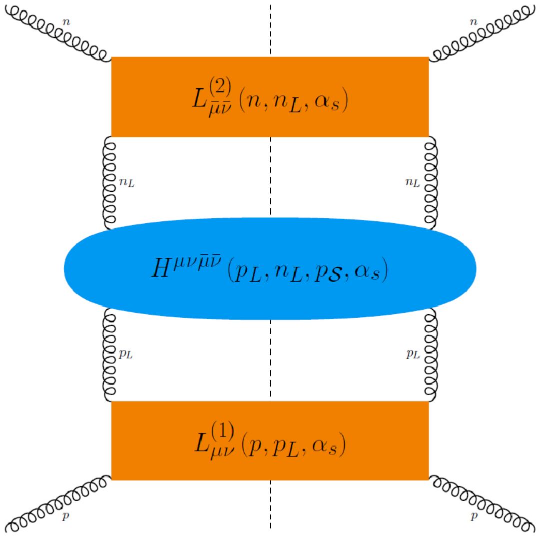
$$H_{ij}(\alpha_s) = [C_i(N, \mathbf{b})C_i(N, \mathbf{b}) + G_i(N, \mathbf{b})G_j(N, \mathbf{b})]$$

$C$, $G$ UNIVERSAL (DEP. ON PARTON)

## APPLIED TRANSVERSE MOMENTUM RESUMMATION

- RESUMMATION REQUIRED FOR ANY $p_t$-DEPENDENT OBSERVABLES WHENEVER $p_t \lesssim M$: E.G. HIGGS, TOP, $W$, $Z$ MOMENTUM DISTRIBUTION, OR HIGGS+JET &C
- MATCHED TO FIXED ORDER
- THRESHOLD RESUMMATION FOLLOWS FROM TRANSVERSE MOMENTUM RESUMMATION
- CLASSIC APPLICATION TO MATCHED NLL+FO (FONLL) HEAVY QUARK PRODUCTION
- COLORED FINAL STATES $\to$ MORE COMPLICATED STRUCTURE
- REQUIRED FOR JET-VETOED CROSS-SECTIONS
- USED IN $q_t$-SUBTRACTION AS A TOOL FOR HIGHER FO COMPUTATIONS
- NNLL AVAILABLE FOR MOST LHC PROCESSESS

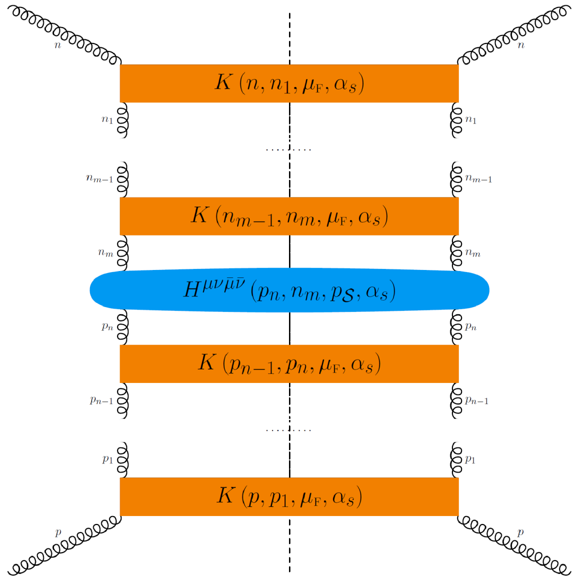
# HIGH-ENERGY RESUMMATION: FACTORIZATION




$$\sigma\left(\frac{Q^2}{s}, \frac{\mu_f^2}{Q^2}, \frac{\mu_r^2}{Q^2}\right) = \int \frac{Q^2}{2s} H^{\mu\nu\bar\mu\bar\nu}\left(n_L, p_L, \Omega_{\mathcal{S}}, \mu_r^2, \mu_f^2, \alpha_s\right) L_{\mu\nu}\left(p_L, p, \mu_r^2, \mu_f^2, \alpha_s\right) L_{\bar\mu\bar\nu}\left(n_L, n, \mu_r^2, \mu_f^2, \alpha_s\right)[dp_L][dn_L]$$

- SUDAKOV PARM. FOR $p_L$, $n_l$; HIGH ENERGY LIMIT: $z \ll 1$, $\frac{k_t^2}{s} \ll 1$
- TWO-GLUON REDUCIBLE DIAGRAM ONLY CONTRIBUTES IN HIGH ENERGY LIMIT, UP TO POWER SUPPRESSION
- REDUCIBLE LINE CONNECT HARD 2GI TO EXTERNAL LADDER
  $\Rightarrow$ RADIATION FROM EXTERNAL LINES AGAIN
- LORENTZ DECOMPOSITION OF LADDERS & HARD:
  ONLY LONGITUDINAL CONTRIBUTES IN HIGH ENERGY LIMIT

# THE LADDER EXPANSION




$$\sigma^{n,m}\left(N, \frac{\mu_f^2}{Q^2}, \alpha_s; \epsilon\right) = \int_0^\infty \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,n}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{dk_{t\,n}^2}{{k_{t\,n}^2}^{1+\epsilon}} \times \int_0^\infty \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,m}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\bar{k}_{t\,m}^2}{{\bar{k}_{t\,n}^2}^{1+\epsilon}}$$

$$\times\, C\left(N, {k_t}^2, {\bar{k}_t}^2, \alpha_s; \epsilon\right)$$

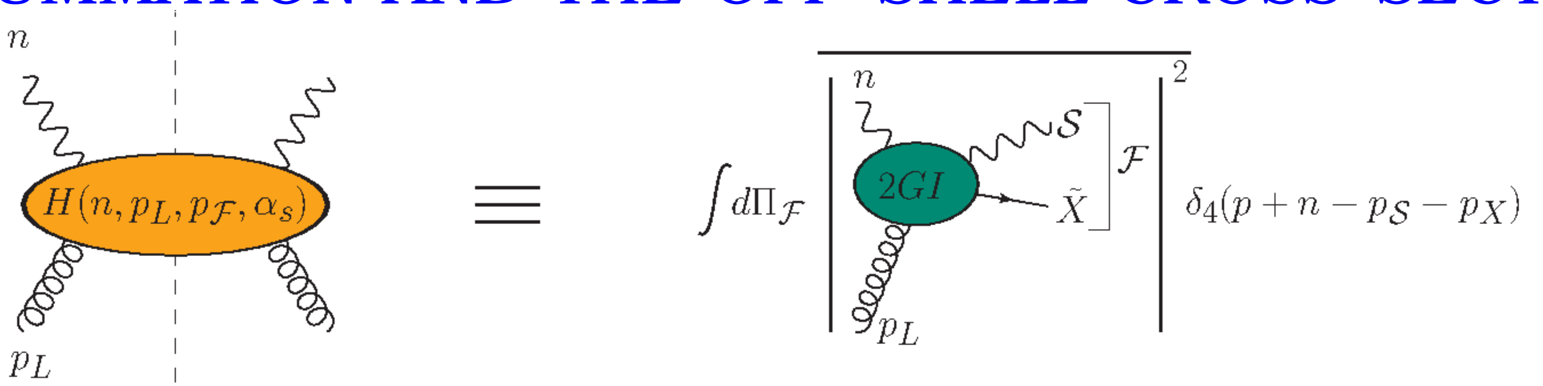
$$\times \int_0^{k_{t\,n}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,n-1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{dk_{t\,n-1}^2}{{k_{t\,n-1}^2}^{1+\epsilon}} \times \cdots \times \int_0^{k_{t\,2}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{k_{t\,1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{dk_{t\,1}^2}{{k_{t\,1}^2}^{1+\epsilon}}$$

$$\times \int_0^{\bar{k}_{t\,m}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{\bar{k}_{t\,m-1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\bar{k}_{t\,m-1}^2}{{\bar{k}_{t\,m-1}^2}^{1+\epsilon}} \times \cdots \times \int_0^{\bar{k}_{t\,2}^2} \left[\gamma\left(N, \left(\frac{\mu_f^2}{\bar{k}_{t\,1}^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\bar{k}_{t\,1}^2}{{\bar{k}_{t\,1}^2}^{1+\epsilon}}.$$

- LADDER OBTAINED BY ITERATION OF A 2GI KERNEL
- THE INTEGRATED KERNEL IS A LL$x$ ANOMALOUS DIMENSION:

$$K\left(N, \left(\frac{\mu_f^2}{{k_t}^2}\right)^\epsilon, \alpha_s; \epsilon\right) = \gamma\left(N, \left(\frac{\mu_f^2}{{k_t}^2}\right)^\epsilon, \alpha_s; \epsilon\right)$$

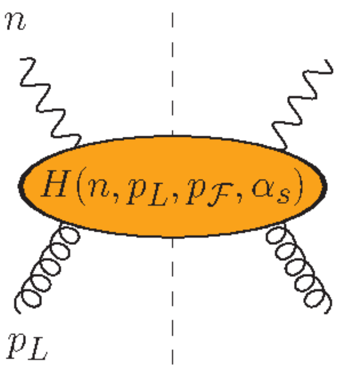
# RESUMMATION AND THE OFF-SHELL CROSS-SECTION



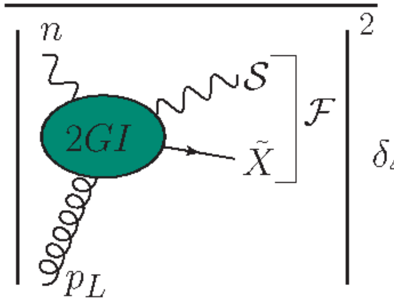

$$\sigma_{\rm res}\left(N,\alpha_s\right) = \gamma\left(\frac{\alpha_s}{N}\right)^2 R\left(\frac{\alpha_s}{N}\right)^2 \int_0^\infty dk_t{}^2\ \left(k_t{}^2\right)^{\gamma\left(\frac{\alpha_s}{N}\right)-1} \int_0^\infty d\bar{k}_t^2\ \left(\bar{k}_t^2\right)^{\gamma\left(\frac{\alpha_s}{N}\right)-1} C\left(N, k_t{}^2, \bar{k}_t^2, \alpha_s\right)$$

- THE ITERATED KERNEL (ANOMALOUS DIMENSION) EXPONENTIATES: $\exp\left[\gamma \ln k_t^2\right] = \left(k_t^2\right)^\gamma$
- THE CONVOLUTIONS LOOK LIKE $k_t$-SPACE MELLIN-TRANSFORMS ($k_t^2$ GLUON OFF-SHELLNESS)
- .

# RESUMMATION AND THE OFF-SHELL CROSS-SECTION




$\equiv$ $\int d\Pi_{\mathcal{F}}$




$\delta_4(p + n - p_{\mathcal{S}} - p_X)$

$$\sigma_{\rm res}\left(N, \alpha_s\right) = h\left(N, \gamma\left(\tfrac{\alpha_s}{N}\right), \gamma\left(\tfrac{\alpha_s}{N}\right), \alpha_s\right)$$

$$h\left(N, M_1, M_2, \alpha_s\right) = M_1 M_2\, R\left(M_1\right) R\left(M_2\right) \int_0^\infty dk_t{}^2 \left(k_t^2\right)^{M_1-1} \int_0^\infty d\bar{k}_t^2 \left(\bar{k}_t^2\right)^{M_2-1} C\left(N, k_t{}^2, (\bar{k}_t)^2, \alpha_s\right)$$

- THE ITERATED KERNEL (ANOMALOUS DIMENSION) EXPONENTIATES
- THE CONVOLUTIONS LOOK LIKE $k_t$-SPACE MELLIN-TRANSFORMS ($k_t^2$ GLUON OFF-SHELLNESS)
- RESUMMATION $\Leftrightarrow$ OFF-SHELL CROSS-SECTION WITH $M = \gamma$

# RESUMMED EVOLUTION

## DUALITY OF THE ANOMALOUS DIMENSIONS

THE ALTARELLI-PARISI EQN IS AN INTEGRO–DIFFERENTIAL EQUATION $\Rightarrow$ IT CAN BE EQUIVALENTLY VIEWED AS $Q^2$–EVOLUTION EQUATION FOR $x$–MOMENTS (usual RG eqn.), OR $x$–EVOLUTION EQUATION FOR $Q^2$–MOMENTS(BFKL eqn.)

EVOLUTION IN $t = \ln Q^2$

$$\frac{d}{dt} G(N,t) = \gamma(N,\alpha_s)\; G(N,t)$$

MELLIN $x$–MOMENTS

$$G(N,t) = \int_0^\infty d\xi\, e^{-N\xi}\; G(\xi,t)$$

EVOLUTION IN $\xi = \ln 1/x$

$$\frac{d}{d\xi} G(\xi,M) = \chi(M,\alpha_s)\; G(\xi,M)$$

MELLIN $Q^2$–MOMENTS

$$G(\xi,M) = \int_{-\infty}^\infty dt\, e^{-Mt}\; G(\xi,t)$$

THE TWO EQUATIONS HAVE THE SAME SOLUTIONS PROVIDED THE EVOLUTION KERNELS ARE RELATED BY

$$\boxed{\begin{aligned}\chi(\gamma(N,\alpha_s),\alpha_s) &= N\\ \gamma(\chi(M,\alpha_s),\alpha_s) &= M\end{aligned}}$$

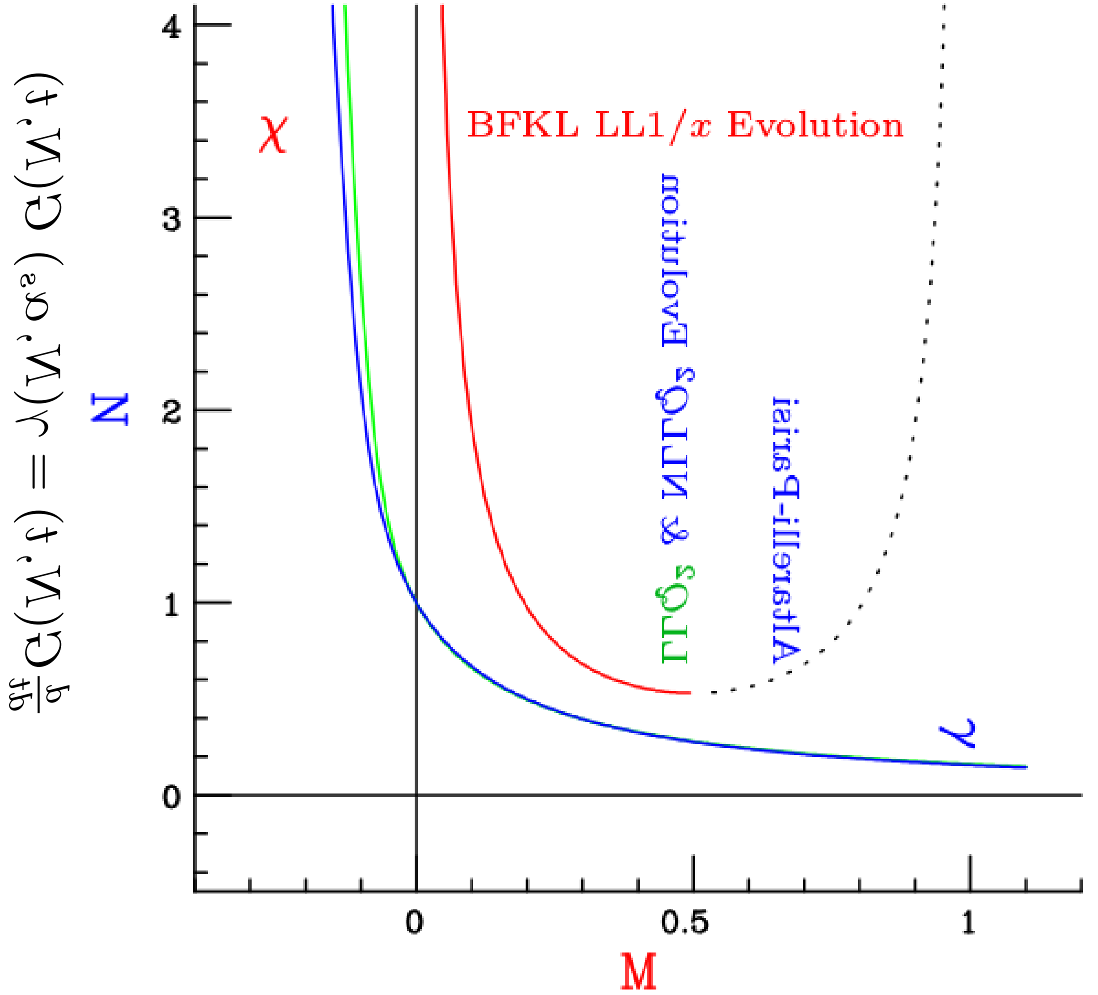
& BOUNDARY CONDITIONS RELATED BY

$$H_0[M] \to G_0(N) = H_0[\gamma(N,\alpha_s)]/\chi'(\gamma(N,\alpha_s))$$

. . . CAN SWITCH FROM LL$Q^2$ TO LL$1/x$

CHOOSING THE EVOLUTION KERNEL

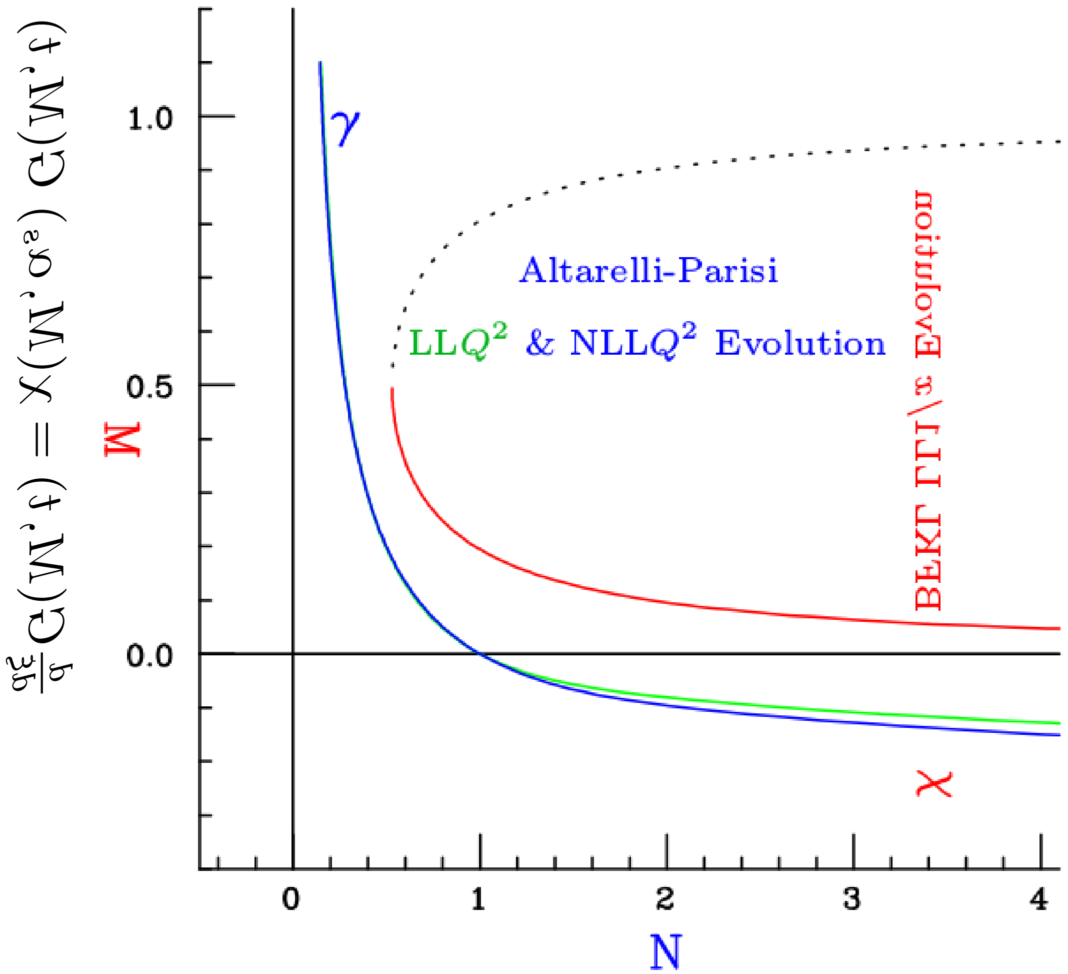
# $\ln 1/x$ EVOLUTION




$$\frac{d}{d\xi} G(M,t) = \chi(M,\alpha_s)\ G(M,t)$$

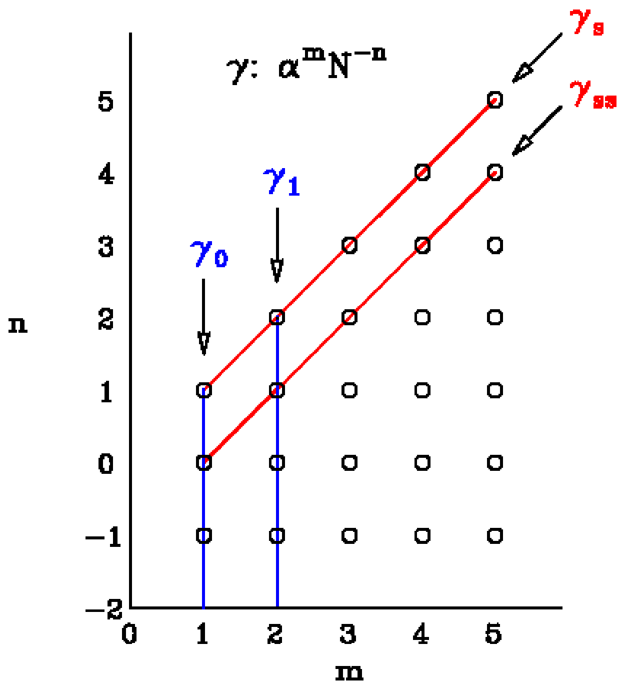
## . . . IN EITHER EQUATION!

# $\ln Q^2$ EVOLUTION

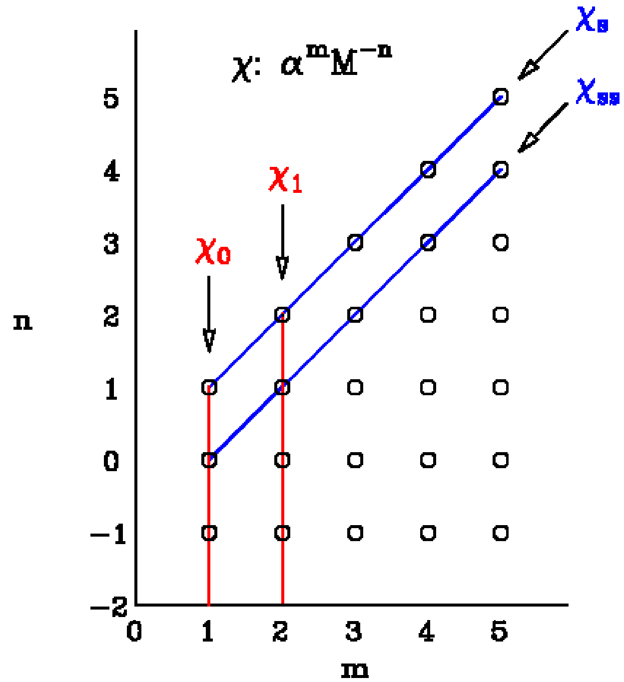



$$\frac{d}{dt} G(N,t) = \gamma(N,\alpha_s)\; G(N,t)$$

# DUAL PERTURBATIVE EXPANSIONS




$\ln Q^2$ EVOLUTION




$\ln 1/x$ EVOLUTION

$$\gamma(N) = \alpha\left(\frac{c^{(1)}_{-1}}{N} + c^{(1)}_0 + \ldots\right) + \alpha^2\left(\frac{c^{(2)}_{-2}}{N^2} + \frac{c^{(2)}_{-1}}{N} + \ldots\right)$$

$$\chi(M) = \alpha\left(\frac{\tilde{c}^{(1)}_{-1}}{M} + \tilde{c}^{(1)}_0 + \ldots\right) + \alpha^2\left(\frac{\tilde{c}^{(2)}_{-2}}{Q^2} + \frac{\tilde{c}^{(2)}_{-1}}{M} + \ldots\right)$$

$$\gamma_s(N) = c^{(1)}_{-1}\frac{\alpha}{N} + c^{(2)}_{-2}\frac{\alpha^2}{N^2} + \ldots$$

$$\chi_s(M) = \tilde{c}^{(1)}_{-1}\frac{\alpha}{M} + \tilde{c}^{(2)}_{-2}\frac{\alpha^2}{Q^2} + \ldots$$

$1/N$ POLES $\Leftrightarrow$ $\ln 1/x$

$1/M$ POLES $\Leftrightarrow$ $\ln Q^2$

$$\gamma_0(N) \Longleftrightarrow \chi_s(\alpha_s/M)$$

$$\gamma_s(\alpha_s/N) \Longleftrightarrow \chi_0(M)$$

# APPLIED HIGH-ENERGY RESUMMATION

- HIGH-ENERGY $\chi$ EVOLUTION KERNEL IS KNOWN UP TO NLO $\Rightarrow$ NLLx EVOLUTION
- HIGH-ENERGY FACTORIZATION HOLDS FOR RAPIDITY & TRANSVERSE-MOMENTUM DISTRIBUTIONS BUT ONLY ESTABLISHED TO LLx
- OFF-SHELL CROSS-SECTION KNOWN FOR DIS, HEAVY QUARKS, HIGGS, DRELL-YAN, JETS
- ACCURATE PHENOMENOLOGY REQUIRES FORMALLY SUBLEADING CORRECTIONS TO NLLx EVOLUTION:
  LL$Q^2$ RUNNING COUPLING & DOUBLE-LEADING $\chi$ SYMMETRIZATION
- ONLY RECENTLY USED FOR IMPROVEMENT & PHENOMENOLOGY

# SUMMARY

- LARGE LOGS $\Leftrightarrow$ SOFT/HARD GLUON EMISSION FROM EXTERNAL LINES
- RADIATION IS EIKONAL $\leftrightarrow$ UNIVERSAL FACTOR
- FACTORIZATION OF EMISSION & HARD PART
- SOFT-COLLINEAR: EXPONENTIATION FROM RG IMPROVEMENT
  "PHYSICAL" ANOMALOUS DIMENSION DETERMINED TO ALL ORDERS
  DETERMINES CROSS-SECTION
- HIGH-ENERGY: EXPONENTIATION FROM KERNEL ITERATION
  KERNEL DETERMINED BY COMPLEMENTARY EVOLUTION EQUATION,
  HARD CROSS-SECTION FROM OFF-SHELL PROCESS