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QCD

AT THE

LARGE HADRON COLLIDER

STEFANO FORTE
UNIVERSITÀ DI MILANO & INFN



UNIVERSITÀ DEGLI STUDI DI MILANO
DIPARTIMENTO DI FISICA



XLIX Herbstschule für
Hochenergiephysik

Maria Laach, 11-14 September 2017

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- THE RENORMALIZATION GROUP AND THE CALLAN-SYMANZIK EQUATION
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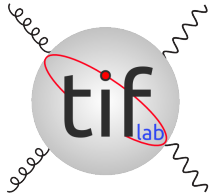
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QCD AT THE LHC

I: FACTORIZATION

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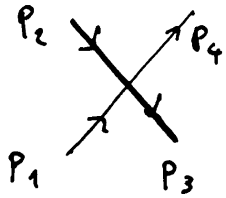
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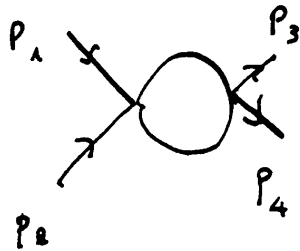
RENORMALIZATION

THE BASIC IDEA

$\mathcal{L}_I = -\frac{g}{24}\phi^4$; $\phi\phi \rightarrow \phi\phi$ ELASTIC SCATTERING OF MASSIVE SCALAR FIELDS



$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi} \frac{1}{s}; \quad s = (p_1 + p_2)^2$$



+ 2 more

$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi} \frac{1}{s} F(s, t): \text{ DIVERGES!};$$

$$t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

$$F(s, t) = \lim_{\Lambda \rightarrow \infty} 1 + \frac{g}{32\pi} \left(3 + \int_0^1 dx \ln \frac{M^2(s)}{\Lambda^2} + s \rightarrow t + s \rightarrow u \right); \quad M^2(s) = m^2 - x(1-x)s$$

RENORMALIZATION: EXPRESS A PHYSICAL OBSERVABLE IN TERMS OF OTHER PHYSICAL OBSERVABLES:

WHAT IS THE CHARGE g ? DEFINE g_{phys} FROM $\left. \frac{d\sigma}{d\cos\theta} \right|_{s=\mu^2} = \frac{g_{\text{phys}}^2}{128\pi} \frac{1}{\mu^2} :$

$$\frac{d\sigma}{d\cos\theta} = \frac{g_{\text{phys}}^2}{128\pi} \frac{1}{s} F(s, t); \quad F(s, t) = 1 + \frac{g_{\text{phys}}}{32\pi} \left(\int_0^1 \ln \frac{M^2(s)}{M^2(4\mu^2)} + s \rightarrow t + s \rightarrow u \right)$$

UV SINGULARITY IS UNIVERSAL $\Rightarrow$ REABSORBED IN DEF. OF THE COUPLING

THE RUNNING COUPLING

THE BASIC IDEA

- PHYSICAL RESULTS CANNOT DEPEND ON RENORMALIZATION SCALE μ_R
- HIGH-ENERGY $\Rightarrow \sigma = \sigma \left(g_{\text{phys}}(\mu_R), \frac{s}{\mu_R^2}, s \right)$
 $\Rightarrow \mu_R \frac{d}{d\mu_R} \sigma = 0$ **RENORMALIZATION GROUP EQUATION**
- **DEPENDENCE** OF $g_{\text{phys}}^2(s)$ ON s **UNIVERSAL**, PERTURBATIVELY COMPUTABLE:
 $\mu_R^2 \frac{d}{d\mu_R^2} g_{\text{phys}}(\mu_R) = \beta(g_{\text{phys}}(\mu_R))$
- SOLUTION: $\sigma = \sigma(g_{\text{phys}}(s), s)$; DEP. ON s FIXED BY DIMENSIONAL ANALYSIS

ϕ^4 THEORY

- HE TOTAL CROSS-SECTION: $\sigma = \frac{g_{\text{phys}}^2(s)}{128\pi} \frac{1}{s}$
- BETA FUNCTION: $\beta(g) = -\beta_0 \frac{1}{2\pi} g^2 + O(g^3)$; $\beta_0 = -\frac{3}{16\pi}$
- **RUNNING COUPLING**: $g(s) = \frac{g \mu_R^2}{1 + \beta_0 \frac{\mu_R^2}{2\pi} \ln \frac{s}{\mu_R^2}} \Rightarrow$ **GROWS W. ENERGY** (CHARGE SCREENING)
- μ^R **DEP. MUST CANCEL** ORDER BY ORDER IN α_s , RESUMMED INTO RUNNING g

QCD & ASYMPTOTIC FREEDOM

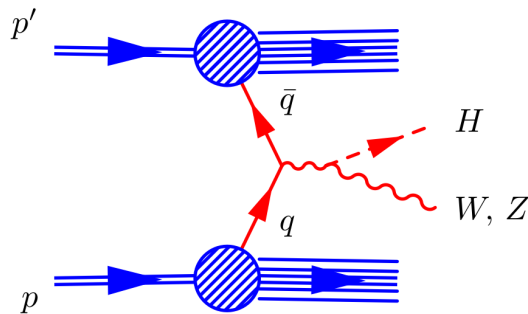
- LAGRANGIAN $\mathcal{L} = -\frac{1}{4}G_{\mu\mu}^a G_{\mu\nu}^a + \sum_i^{n_f} \bar{\psi}_i(i\not{D} - m_i)\psi_i$;
 $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}g_s A_\mu^b A_\nu^c$; $D_\mu = \partial_\mu - ig_s \lambda^a A_\mu^a$; λ^a GENERATORS OF SU(3); $[\lambda^a, \lambda^b] = if^{abc}\lambda^c$ ACTING ON ψ
- PERTURBATIVE EXPANSION PARAMETER $\alpha_s \equiv \frac{g_s^2}{4\pi}$
- BETA FUNCTION: $\beta(\alpha_s) = -\beta_0 \frac{1}{2\pi} \alpha_s^2 + O(\alpha_s^3)$; $\beta_0 = \frac{1}{6} (11C_A - 2n_f)$;
for $SU(N_c)$, $C_A = N_c$
- DO NOT CONFUSE N_c GAUGE GROUP & n_f NUMBER OF FERMIONS (QUARKS)
- RUNNING COUPLING: $\alpha(s) = \frac{\alpha_{\mu^2}}{1 + \beta_0 \frac{\alpha_{\mu^2}}{2\pi} \ln \frac{s}{\mu^2}} \Rightarrow$ DECREASES AT HIGH ENERGY:

ASYMPTOTIC FREEDOM

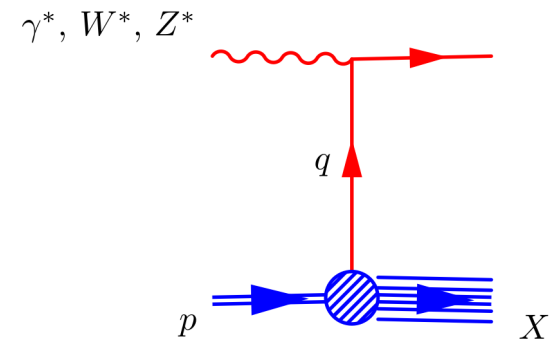
PARTON MODEL

- ASSUME AT HIGH-ENERGY **INTERACTIONS** DESCRIBED BY **FREE-FIELD** THEORY
- FUNDAMENTAL **SCATTERING** HAPPENS **AMONG CONSTITUENTS** “PARTONS” (QUARKS & GLUONS)
- IF NO RESCATTERING, ALL MOMENTA PARALLEL

HADRON-HADRON



PHOTON (LEPTON) -HADRON



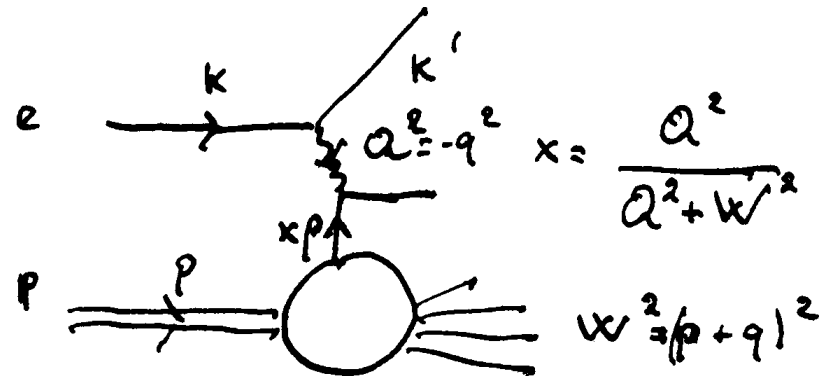
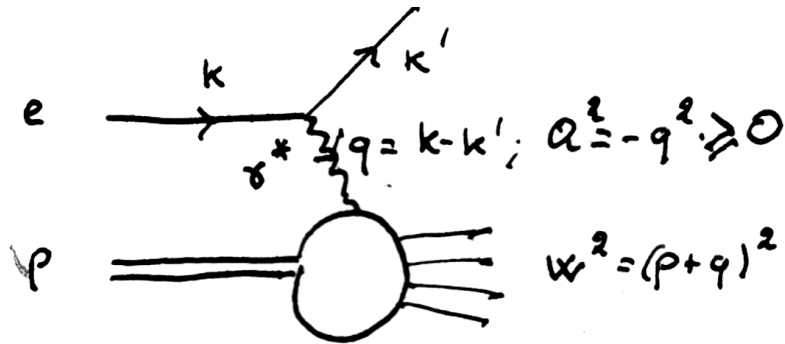
- ONE PARTON PER HADRON: $\hat{p}_1 = x_1 p_1$; $\hat{p}_2 = x_2 p_2$; ($p_i \Rightarrow$ hadrons, $\hat{p}_i \Rightarrow$ partons)
- FOR **HADRONIC PROCESSES**: $\hat{s} = 2x_1 x_2 p_1 \cdot p_2 = x_1 x_2 s$
(HIGH-ENERGY, NEGLECTING ALL MASSES)
- FOR **LEPTON-HADRON**: $2xp \cdot q = x [(p + q)^2 - q^2] = xW^2 + Q^2$ ($xp \Rightarrow$ parton, q photon)
 $W^2 = (p + q)^2 \equiv s_{\gamma^* p}$, $Q^2 = -q^2 \geq 0$, $x_b \equiv \frac{2p \cdot q}{Q^2}$ (Bjorken x) $\Rightarrow x = x_b$ IF $W^2 = 0$
(TREE-LEVEL PROCESS)

DEEP-INELASTIC $e - p$ SCATTERING

COLLINEAR SINGULARITIES

- LEPTON-PROTON REDUCED TO PHOTON (W , Z)-QUARK; $k' - k = q$
- CROSS-SECTION DEPENDS ON MOMENTA (p , q)
 $\Rightarrow$ **SCALE** $Q^2 = -q^2$ & **DIMENSIONLESS RATIO** $x_B = \frac{Q^2}{2p \cdot q}$ (Bjorken variable)
- **LEADING-ORDER**: $(xp + q)^2 = 0 \Rightarrow x = x_B$; CROSS-SECTION $\sigma \propto \delta(x - x_B)$

LEADING-ORDER

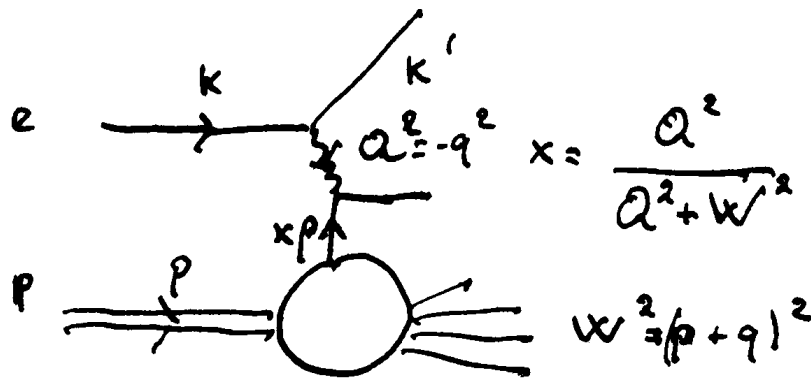


COLLINEAR SINGULARITIES

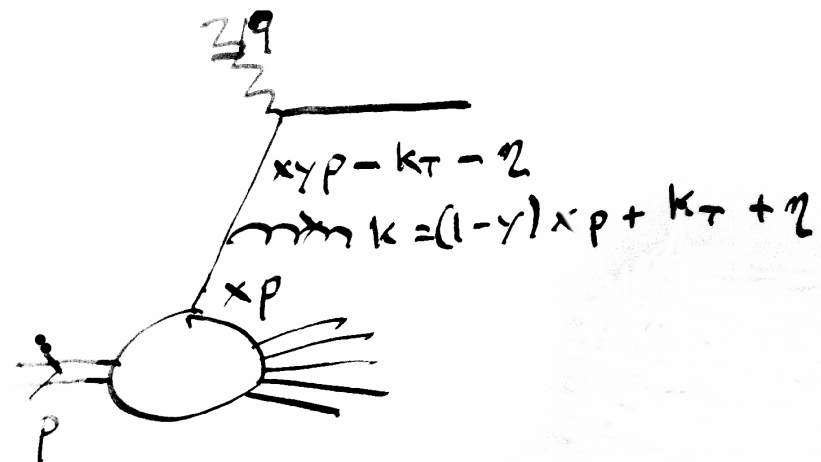
DEEP-INELASTIC $e - p$ SCATTERING

- CROSS-SECTION DEPENDS ON MOMENTA (p, q)
 $\Rightarrow$ **SCALE** $Q^2 = -q^2$ & **DIMENSIONLESS RATIO** $x_B = \frac{Q^2}{2p \cdot q}$ (Bjorken variable)
- LEADING-ORDER**: $(xp + q)^2 = 0 \Rightarrow x = x_B$; CROSS-SECTION $\sigma \propto \delta(x - x_N)$
- NEXT-TO-LEADING-ORDER**: PARAMETRIZE MOMENTUM OF EMITTED QUARK
 $k = (1 - y)xp + k_T + \eta$ SUCH THAT $k^2 = p^2 = \eta^2 = p \cdot k_t = \eta \cdot k_t = 0$, INTEGRATE
INCLUDING PROPAGATOR $\frac{d^3k}{E} \frac{1}{k^2} = dz \frac{d^2k_T}{k_t^2}$ **LOG DIVERGENT** AS $k_T \rightarrow 0$ (COLLINEAR KINEMATICS)

LEADING-ORDER



NEXT-TO-LEADING-ORDER

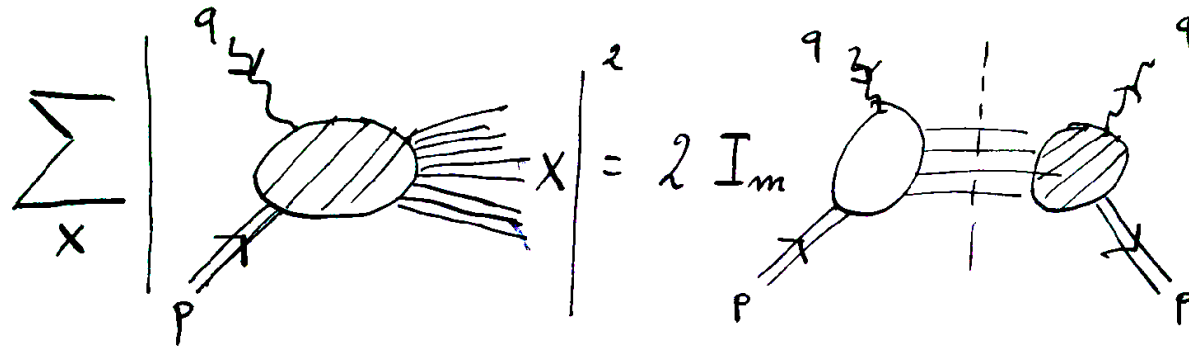


THE WILSON EXPANSION

DEEP-INELASTIC $e - p$ SCATTERING

OPTICAL THEOREM FOR THE INCLUSIVE CROSS SECTION

$$\sigma = \sum_X |\langle X | p\gamma^* \rangle|^2 = \frac{2}{\pi} \text{Im} \langle p\gamma^* | p\gamma^* \rangle$$



OPERATOR-PRODUCT MATRIX ELEMENT

$$\langle p\gamma^*(q) | p\gamma^*(q) \rangle = i \int \frac{d^4x}{(2\pi)^4} e^{iqx} \langle p | J^\mu(x) J^\nu(0) | p \rangle \equiv W^{\mu\nu}$$

OPERATOR-PRODUCT EXPANSION

$$J^\mu(x) J^\nu(0) = \sum_i C_i(x^2) O_i^{\mu\nu} = \sum_n \sum_k C_{nk}(x^2) O_{nk}^{\mu\nu\alpha_1 \dots \alpha_k} x_{\alpha_1} \dots x_{\alpha_k}$$

THE WILSON EXPANSION LEADING-TWIST FACTORIZATION

DIMENSIONAL ANALYSIS: $C_{nk}(x^2) \sim (x^2)^{\frac{d_n}{2} - k - d_j}$; d_n, d_j MASS DIM. OF O_{nk} , $J^\mu \Rightarrow$ AS $x^2 \rightarrow 0$,

$$J^\mu(x)J^\nu(0) = \sum_{k, \min[d_n - k]} C_{nk}(x^2) O_{nk}^{\mu\nu\alpha_1 \dots \alpha_{k-2}} x_{\alpha_1} \dots x_{\alpha_k} + O(x^2 \Lambda_p^2)$$

Λ_p : CHARACTERISTIC SCALE OF QCD PROTON MATRIX ELEMENTS;

$d_n - k$ **TWIST** OF THE OPERATOR = dim.-spin

TWIST 2 OPERATORS

$$O_k^{(2, q)\mu \dots \alpha_{k-2}} = \bar{\psi} \gamma^\mu \partial^\nu \partial^{\alpha_1} \dots \partial^{\alpha_{k-2}} \psi; \quad O_k^{(2, g)\mu \dots \alpha_{k-2}} = G_{\mu\alpha} \partial^{\alpha_1} \dots \partial^{\alpha_{k-2}} G^\nu{}_\alpha$$

- OPERATOR BASIS: SYMMETRIZE AND SUBTRACT TRACE (SPIN EIGENSTATES)
- BEYOND LEADING ORDER $\partial^\mu \rightarrow D^\mu$ (GAUGE INVARIANCE)
- COLOR INDICES SUMMED: ONLY COLOR-SINGLET OPERATORS
- ONE FERMION OPERATOR FOR EACH QUARK FLAVOR

OPE MATRIX ELEMENTS

$$\begin{aligned} i \int \frac{d^4 x}{(2\pi)^4} e^{iqx} \langle p | J^\mu(x) J^\nu(0) | p \rangle &= \sum_k \langle p | O^{(2, q)\mu \dots \alpha_{k-2}} | p \rangle i \int \frac{d^4 x}{(2\pi)^4} e^{iqx} C_k^{(2)}(x^2) x_{\alpha_1} \dots x_{\alpha_k} \\ &= \sum_k \left(\frac{2p \cdot q}{Q^2} \right)^{k-2} \frac{p^\mu p^\nu}{Q^2} A_K C_k(Q^2) + \dots \end{aligned}$$

- $\langle p | O^{(2, q)\mu \nu \alpha_1 \dots \alpha_{k-2}} | p \rangle = A_k 2p^\mu p^\nu p^{\alpha_1} \dots p^{\alpha_{k-2}}$; A_k reduced mat. el.
- $i \int \frac{d^4 x}{(2\pi)^4} e^{iqx} C(x^2) x_\alpha = \frac{2q^\alpha}{Q^2} C(Q^2)$ etc.
- TENSOR STRUCTURE SYMMETRIZED & FIXED BY CURRENT CONSERVATION

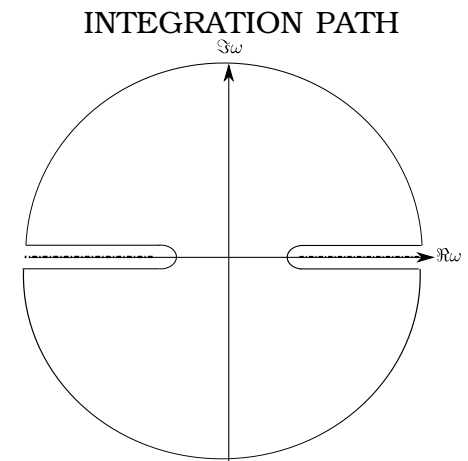
FACTORIZATION: FROM MOMENT SPACE TO PHYSICAL SPACE

$$W^{\mu\nu}(x, Q^2) = \sum_k \left(\frac{1}{x}\right)^{k-2} \frac{p^\mu p^\nu}{Q^2} A_K C_k(Q^2) + \dots$$

- PARAMETRIZE $W_{\mu\nu}$ W. FORM FACTORS W_i :

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1 + \left(p^\mu - q^\mu \frac{p \cdot q}{q^2}\right) \left(p^\nu - q^\nu \frac{p \cdot q}{q^2}\right) \frac{2x}{Q^2} W_2$$
- $W_2(x, Q^2) = \sum_k \omega^{k-1} A_k C_k(Q^2)$
 $\Rightarrow A_k C_k(Q^2) = \oint \frac{d\omega}{2\pi i} \omega^{-k} W_2(\omega, Q^2); \omega \equiv \frac{1}{x}$
- DEFORM THE INTEGRATION PATH

$$\begin{aligned} A_k C_k(Q^2) &= 2 \int_1^\infty \frac{d\omega}{2\pi i} \omega^{-k} \left(W_2(x\omega + i\epsilon, Q^2) - W_2(x\omega - i\epsilon, Q^2) \right) \\ &= \frac{2}{\pi} \int_0^1 dx x^{k-2} \text{Im} W_2(x, Q^2) = \int_0^1 dx x^{k-2} F_2(x, Q^2) \end{aligned}$$



$$F_i(x, Q^2) = \frac{2}{\pi} \text{Im} W_i(x, Q^2):$$

STRUCTURE FUNCTION (FORM-FACTOR OF XSECT.)

MELLIN TRANSFORMS & CONVOLUTIONS

$$h(x) = \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y) \Leftrightarrow h_n = g_n f_n; \quad f_n = \int_0^1 dx x^{n-1} f(x), \text{ SIMILAR FOR } h, g$$

FACTORIZATION IN PHYSICAL SPACE

$$F_2(x, Q^2) = x \int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, Q^2\right) q(y, Q^2); \quad \int_0^1 dx x^{n-1} C(x, Q^2) = C_n(Q^2); \quad \int_0^1 dx x^{n-1} q(x) = A_n$$

COEFFICIENT FUNCTIONS AND PARTON DISTRIBUTIONS

Q: WHAT IS THE MEANING OF $q(x)$, $C(x)$?

A: COMPUTE PERTURBATIVELY!

- TAKE MATRIX ELEMENT IN FREE QUARK STATE: $\langle p | O^{(2, q) \mu \dots \alpha_{k-2}} | p \rangle = 2p^\mu \dots p^{\alpha_{k-2}} \Rightarrow A_k = 1$
EXAMPLE: $k = 2 \Rightarrow$ ENERGY-MOMENTUM TENSOR:
 A_2 FRACTION OF THE QUARK ENERGY CARRIED BY ITSELF (!)
- IN PROTON, A_2 IS FRACTION OF **PROTON** ENERGY CARRIED BY QUARK
SIMILARLY FOR GLUON OPERATORS, OTHER FLAVORS
- $A_k = 1$ FOR ALL $k \Leftrightarrow q(x) = \delta(1 - x)$

COEFFICIENT FUNCTION & PARTONIC CROSS-SECTION

IN A **QUARK** STATE, $F_2(x, Q^2) = x \int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, Q^2\right) \delta(1 - y) = x C(x, Q^2) \equiv \hat{\sigma}(x) \Leftrightarrow$

$C(x)$: STRUCTURE FUNCTION (**CROSS-SECTION**) COMPUTED WITH A **QUARK TARGET**:
PARTONIC CROSS-SECTION

OPERATOR MATRIX ELEMENTS & PARTON DISTRIBUTIONS

IN A **PROTON** STATE $\sigma(x) \equiv \frac{F_2(x)}{x} = \int_x^1 \frac{dy}{y} \hat{\sigma}\left(\frac{x}{y}, Q^2\right) q(y) = \int_x^1 \frac{dy}{y} \hat{\sigma}\left(\frac{Q^2}{y^2 p \cdot q}, Q^2\right) q(y)$

$q(x)$: **RELATIVE WEIGHT** OF THE CONTRIBUTION THE CROSS SECTION FROM INITIAL-STATE **QUARKS**
WITH MOMENTUM FRACTION x : **PARTON DISTRIBUTION (PDF)**

NOTE NOT A PROBABILITY (BEYOND LO), NEGATIVE WEIGHTS ALLOWED!

FACTORIZATION: LEPTON-HADRON VS HADRON HADRON ONE HADRON IN THE INITIAL STATE

$$\sigma(x) = \int dz \int_x^1 dy \delta(x - yz) q(y) \hat{\sigma}(z) = \int_x^1 \frac{dy}{y} q(y) \hat{\sigma}\left(\frac{x}{y}\right) = \int_x^1 \frac{dy}{y} q\left(\frac{x}{y}\right) \hat{\sigma}(y) = [\sigma \otimes q](x)$$

- $x = x_B$: SCALING VARIABLE FOR HADRONIC PROCESS (MEASURED HADRON KINEMATICS)
- z : SCALING VARIABLE FOR PARTONIC PROCESS (COMPUTED PARTONIC FEYNMAN DIAGRAM)
- y : MOMENTUM FRACTION CARRIED BY INCOMING PARTON
- Q^2 DEP. OF σ , $\hat{\sigma}$ & q OMITTED

TWO HADRONS IN THE INITIAL STATE

$$\begin{aligned} \sigma(\tau) &= \int dz \int_x^1 dx_1 dx_2 \delta(\tau - x_1 x_2 z) q_1(x_1) q_2(x_2) \hat{\sigma}(z) = \int_\tau^1 \frac{dx_1}{x_1} \int_\tau^1 \frac{dx_2}{x_2} dy q_1(x_1) q_2(x_2) \hat{\sigma}\left(\frac{\tau}{x_1 x_2}\right) \\ &= \int_\tau^1 \frac{dy}{y} \mathcal{L}(y) \hat{\sigma}\left(\frac{\tau}{y}\right) = [\mathcal{L} \otimes \hat{\sigma}](\tau) \\ \mathcal{L}(y) &\equiv \int_y^1 \frac{dx_1}{x_1} q_1(y) q_2\left(\frac{y}{x_1}\right) = [q_1 \otimes q_2](y) \end{aligned}$$

- τ : SCALING VARIABLE FOR HADRONIC PROCESS (MEASURED HADRON KINEMATICS)
- z : SCALING VARIABLE FOR PARTONIC PROCESS (COMPUTED PARTONIC FEYNMAN DIAGRAM)
- x_1, x_2 : MOMENTUM FRACTIONS CARRIED BY INCOMING PARTONS
- $\mathcal{L}$: PARTON LUMINOSITY

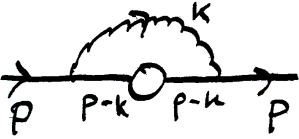
COLLINEAR SINGULARITIES AND THE RENORMALIZATION GROUP

WHAT HAPPENED TO THE **COLLINEAR SINGULARITIES**?
 WHAT HAPPENS **BEYOND LEADING ORDER**?

RENORMALIZATION OF OPERATOR MATRIX ELEMENTS

$$A_k 2p^{\mu_1} \dots p^{\mu_n} = \langle p | O^{(2,q)\mu_1 \dots \mu_n} | p \rangle = \langle p | \bar{\psi} \gamma^\mu D^{\mu_1} \dots D^{\mu_n} \psi | p \rangle$$

LEADING ORDER:  $= \gamma^{\mu_1} \dots p^{\mu_n} \Rightarrow A_n = 1$
NEXT-TO-LEADING ORDER:

 + 4 more = $\gamma^\mu p^\nu \dots p^\alpha \mu^{2\epsilon} \Gamma[\epsilon] \frac{\alpha_s}{2\pi} \left[1 + 4 \sum_{j=2}^n \frac{1}{j} - \frac{2}{n(n+1)} \right] \Rightarrow$
 $A_n = A_n(\mu^2)$

RENORMALIZE: $A_n^{\text{ren}}(\mu^2) = Z_n^{\text{ren}}(\mu_F^2) A_n(\mu^2) = \frac{A_n(\mu^2)}{A_n(\mu_f^2)}:$

$\mu_F^2 \Rightarrow$ SCALE AT WHICH A QUARK IS A SINGLE QUARK: **FACTORIZATION SCALE**

LOG SCALE DEP.: $\mu_F^2 \frac{d}{d\mu_F^2} Z(\mu_F^2) = -\mu_F^2 \frac{d}{d\mu_F^2} A_n^{\text{ren}}(\mu_F^2) = \mu^2 \frac{d}{d\mu} A_n(\mu^2) \equiv \gamma_N,$

ANOMALOUS DIMENSION INDEPENDENT OF μ_F^2 (DIM. ANALYSIS): $\gamma_n = \alpha_s(\mu_R^2) \gamma_N^{(0)} + O(\alpha_s^2)$
RENORMALIZATION GROUP

- $A_k C_k(Q^2) = \int_0^1 dx x^{k-2} F_2(x, Q^2)$ ARE PHYSICAL OBSERVABLES ($\sigma = \frac{F_2}{x}$ IS)
- CANNOT DEPEND ON μ_F : $\mu_F^2 \frac{d}{d\mu_F^2} A_k C_k = 0$
- $A_N = A_N(\mu_F^2), C_N = C_N \left(\frac{Q^2}{\mu_R^2}, \frac{\mu_F^2}{\mu_R^2}, \alpha(\mu_R) \right)$; LET $\mu_F = \mu_R = \mu$, CAN RELAX $\mu_R = k\mu_F$

CALLAN-SYMANZIK (RENORMALIZATION GROUP) EQUATION

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_n(\alpha(\mu^2)) \right] C_n \left(\frac{Q^2}{\mu^2}, \alpha(\mu^2) \right) = 0$$

ALTARELLI-PARISI EVOLUTION & COEFFICIENT FUNCTIONS VS PARTON DISTRIBUTIONS SOLVING THE RGE

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_n(\alpha(\mu^2)) \right] C_n \left(\frac{Q^2}{\mu^2}, \alpha(\mu^2) \right) = 0$$

- let $\alpha_s = \alpha(Q^2) \Rightarrow Q^2 \frac{d}{dQ^2} C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2) \right) = \gamma_N(\alpha_s(Q^2)) C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2) \right)$
- integrate $\Rightarrow C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2) \right) = C(1, \alpha_s(Q^2)) \exp \int_0^{\ln Q^2} d \ln \mu^2 \gamma_n(\alpha(\mu^2))$
- use α as integration variable $\Rightarrow C_N \left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2) \right) = C(1, \alpha_s(Q^2)) \exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha \frac{\gamma_N(\alpha)}{\beta(\alpha)}$
- recall $C_N \left(\frac{Q^2}{\mu^2} \right) A_N(\mu^2)$ independent of $\mu^2 \Rightarrow$

$$\sigma_n \equiv \int_0^1 dx x^{n-1} \sigma(x, Q^2) = C_n \left(\frac{Q^2}{\mu^2} \right) A_N(\mu^2)$$

$$\sigma_n = C(1, \alpha_s(Q^2)) \left[\exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha \frac{\gamma_N(\alpha)}{\beta(\alpha)} \right] A_n(\mu^2) = \hat{\sigma}_n(1, \alpha_s(Q^2)) A_n(Q^2), \text{ WHERE}$$

$$Q^2 \frac{d}{dQ^2} A_n(Q^2) = \gamma_n(\alpha_s(Q^2)) A_n(Q^2) \Leftrightarrow Q^2 \frac{d}{dQ^2} q(, xQ^2) = [P(\alpha_s(Q^2)) \otimes q(Q^2)](x)$$

ALTARELLI-PARISI EQUATION, $\int_0^1 dx x^{n-1} P(x) = \gamma_N$ SPLITTING FUNCTION

- **POWER COUNTING:** $\hat{\sigma}_n(1, \alpha_s(Q^2)) = \hat{\sigma}_n^{(0)} + \alpha_s(Q^2) \hat{\sigma}_n^{(1)} + \dots; \sigma_n^{(0)}$ “PARTON MODEL”

$$\exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} d\alpha \frac{\gamma_N(\alpha)}{\beta(\alpha)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{\frac{\gamma_n^{(0)}}{\beta_0}} + \dots = \left(1 + \beta_0 \alpha_s(\mu^2) \ln \frac{Q^2}{\mu^2} \right)^{\frac{\gamma_n^{(0)}}{\beta_0}} + O(\alpha^2 \ln)$$

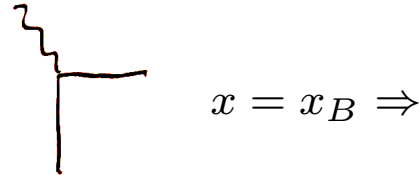
LEADING LOG

$$= 1 + \alpha_s(\mu^2) \gamma_n^{(0)} \ln \frac{Q^2}{\mu^2} + O(\alpha^2) \text{ LEADING ORDER}$$

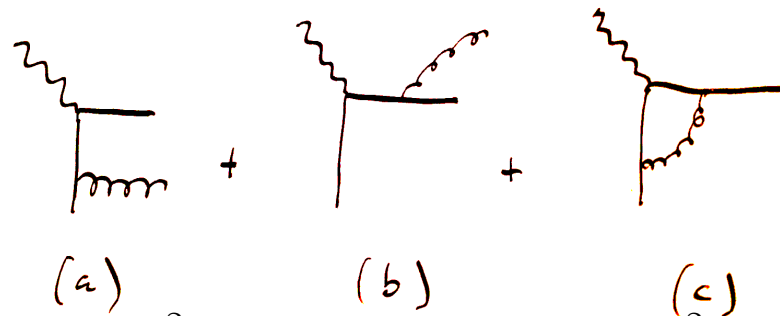
ALTARELLI-PARISI EVOLUTION & FACTORIZATION OF COLLINEAR SINGULARITIES

THE **HADRONIC** CROSS SECTION: $\sigma(Q^2, x) = \hat{\sigma}(\alpha(Q^2)) \otimes q(Q^2) = \hat{\sigma}\left(\frac{Q^2}{\mu_F^2}, \alpha(\mu_F^2)\right) \otimes q(\mu_F^2)$

PARTONIC CROSS SECTION: AT **LEADING ORDER**
 $\hat{\sigma}^{(0)}(x) \propto \delta(1-x) = \Sigma^{(0)}\delta(1-x);$



NEXT-TO-LEADING ORDER, LEADING LOG:



$$\hat{\sigma}^{\text{NLO, LL}}\left(\frac{Q^2}{\mu^2}, x\right) = \hat{\sigma}^{(0)}(1, x) \otimes \alpha_s(Q^2)P(x) \ln \frac{Q^2}{\mu^2} = \Sigma^{(0)}\alpha_s(Q^2)P(x) \ln \frac{Q^2}{\mu^2}$$

WHERE DOES THE LOG COME FROM? DIAGR. (c) $\propto$ **LO** $\Rightarrow \propto \delta(1-x)$; DIAGR. (b) IS FINITE $\Rightarrow$ NO LOG DEPENDENCE; DIAGR. (a) **COLLINEAR DIVERGENCE** HAS BEEN FACTORIZED AND REABSORBED IN PARTON DISTRIBUTION;

NEXT-TO-LEADING ORDER (FULL): $\hat{\sigma}^{\text{NLO}}\left(\frac{Q^2}{\mu^2}, x\right) = \alpha_s(Q^2)\hat{\sigma}^{(1)}(1, x) + \hat{\sigma}^{\text{NLO, LL}}\left(\frac{Q^2}{\mu^2}, x\right)$

COLLINEAR FACTORIZATION

- COLLINEAR EMISSION LEADS TO **UNIVERSAL PREFACTOR** (SPLITTING FUNCTION) CONVOLUTED WITH CROSS-SECTION
- ALTARELLI-PARISI SUMS UP & **FACTORIZES EMISSION** LADDER
- **DIAGRAMMATIC ARGUMENT** DOES NOT RELY ON OPE



PARTON KINEMATICS vs. HADRON KINEMATICS

$$\sigma(\tau) = \int_{\tau}^1 \frac{dy}{y} \mathcal{L}(y) \hat{\sigma}\left(\frac{\tau}{z}\right); \quad \mathcal{L}(y) \equiv \int_y^1 \frac{dx_1}{x_1} q_1(y) q_2\left(\frac{y}{x_1}\right)$$

$$Q^2 \frac{d}{dQ^2} f_i(, xQ^2) = \sum_j \int_x^1 P_{ij} \left(\alpha_s(Q^2), \frac{x}{y} \right) q_j(Q^2)$$

- q_i QUARKS AND GLUONS; GLUON MIXES WITH SINGLET $\Sigma = \sum_i q_i$
- PARTONIC CHANNEL DEPENDS ON PHYSICAL PROCESS (e.g. $W^+ \Rightarrow u\bar{d}$ fusion)
- WHICH PARTON MOMENTUM FRACTIONS CONTRIBUTE TO A GIVEN HADRONIC PROCESS ?

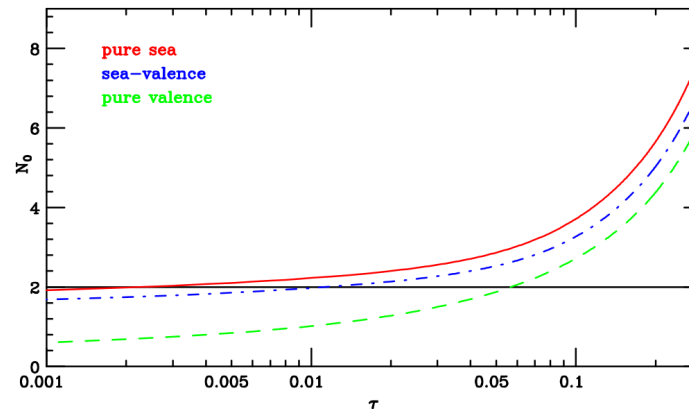
INVERSION OF MELLIN TRANSFORMS

$$f_n = \int_x^1 x^{n-1} f(x) \Leftrightarrow F(x) = \int_{-i\infty}^{+i\infty} x^{-n} f_n$$

integrate to the right of convergence abscissa

- MELLIN INVERSION DOMINATED BY SADDLE POINT
- POSITION OF SADDLE CONTROLLED BY LUMINOSITY
DEPENDENCE ON x OF $\mathcal{L}$ POWERLIKE, OF $\hat{\sigma}$ LOGARITHMIC
- PDF PEAKED AT SMALL $x \Rightarrow$ LUMI PEAKS AT SMALL N

SADDLE VS $\tau = Q^2/s$



SUMMARY

- **INFINITIES** ARISE WHEN EXPRESSING PHYSICAL OBSERVABLES IN TERMS OF ZERO-DISTANCE PHYSICS
RENORMALIZATION REMOVES THEM EXPRESSING THEM IN TERMS OF OTHER PHYSICAL OBSERVABLES
- **FACTORIZATION** OF SHORT- AND LONG DISTANCE PHYSICS RELIES ON SCALE SEPARATION
INTERFERENCE IS POWER-SUPPRESSED
- **COLLINEAR SINGULARITIES** ARISE WHEN INTEGRATING **PERTURBATIVE** HARD CROSS-SECTIONS DOWN TO SOFT SCALES
COLLINEAR FACTORIZATION REABSORBS THEM IN **NON-PERTURBATIVE** INITIAL CONDITIONS
- FACTORIZATION IS MULTIPLICATIVE IN **MELLIN SPACE** WHERE HARD CROSS-SECTIONS ARE ANALYTIC FUNCTIONS
ONE-TO-ONE MAPPING BETWEEN HADRONIC (OBSERVABLE) KINEMATICS AND MELLIN-SPACE VARIABLE IN SADDLE APPROXIMATION