RESUMMATION I

RENORMALIZATION GROUP

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23 August 2021

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 740006

SUMMARY

LECTURE II: RESUMMATION 1

- Collinear singularities
- Universality and factorization
- Asymptotic freedom in parton language
- Singularities and logarithms
- Soft log universality: the eikonal limit
- Renormalization group: Sudakov resummation
- The structure of resummed results: soft logs

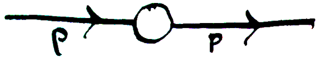
COLLINEAR SINGULARITIES AND RENORMALIZATION

WHAT HAPPENED TO THE COLLINEAR SINGULARITIES?

WHAT HAPPENS BEYOND LEADING ORDER?

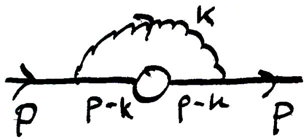
RENORMALIZATION OF OPERATOR MATRIX ELEMENTS

$$A_k 2p^{\mu_1}\dots p^{\mu_n} = \langle p|O^{(2,\,q)\,\mu_1\cdots\mu_n}|p\rangle = \langle p|\bar\psi\gamma^\mu D^{\mu_1}\dots D^{\mu_n}\psi|p\rangle$$

LEADING ORDER: 

$=\gamma^{\mu_1}\dots p^{\mu_n} \Rightarrow A_n = 1$

NEXT-TO-LEADING ORDER:



$$+4\text{ more}= \gamma^\mu p^\nu \dots p^\alpha \mu^{2\epsilon}\Gamma[\epsilon]\frac{\alpha_s}{2\pi}\left[1+4\sum_{j=2}^{n}\frac{1}{j}-\frac{2}{n(n+1)}\right] \Rightarrow$$

$$A_n = A_n(\mu^2)$$

RENORMALIZE: $A_n^{\rm ren}(\mu^2) = Z_n^{\rm ren}(\mu_F^2)A_n(\mu^2) = \frac{A_n(\mu^2)}{A_n(\mu_f^2)}$:

$\mu_F^2 \Rightarrow$ SCALE AT WHICH A QUARK IS A SINGLE QUARK: FACTORIZATION SCALE

LOG SCALE DEP.: $\mu_F^2\frac{d}{d\mu_F^2}Z(\mu_F^2) = -\mu_F^2\frac{d}{d\mu_F^2}A_n^{\rm ren}(\mu_F^2) \equiv \gamma_N$,

ANOMALOUS DIMENSION INDEPENDENT OF μ_F^2 (DIM. ANALYSIS): $\gamma_N = \alpha_s(\mu_R^2)\gamma_N^{(0)} + O(\alpha_s^2)$

RENORMALIZATION GROUP INVARIANCE & RESUMMATION

RENORMALIZATION GROUP

- $A_N C_N(Q^2) = \int_0^1 dx x^{N-2} F_2(x, Q^2)$ ARE PHYSICAL OBSERVABLES ($\sigma = \frac{F_2}{x}$ IS)
- CANNOT DEPEND ON μ_F: $\mu_F^2 \frac{d}{d\mu_F^2} A_N C_N = 0$
- $A_N = A_N(\mu_F^2)$, $C_N = C_N\left(\frac{Q^2}{\mu_R^2}, \frac{\mu_F^2}{\mu_R^2}, \alpha(\mu_R)\right)$; LET $\mu_F = \mu_R = \mu$, CAN RELAX $\mu_R = k\mu_F$

CALLAN-SYMANZIK (RENORMALIZATION GROUP) EQUATION

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha)\frac{\partial}{\partial \alpha} + \gamma_N(\alpha(\mu^2))\right] C_n\left(\frac{Q^2}{\mu^2}, \alpha(\mu^2)\right) = 0$$

SOLVING THE RGE

let $\alpha_s = \alpha(Q^2) \Rightarrow Q^2 \frac{d}{dQ^2} C_N\left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right) = \gamma_N\left(\alpha_s(Q^2)\right) C_N\left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right)$

SOLUTION: $C_N\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = C_N\left(1, \alpha_s(Q^2)\right) \exp \int_{\mu^2}^{Q^2} \frac{d\lambda^2}{\lambda^2} \gamma_N[\alpha(\lambda^2)]$

recall $\alpha_s(Q^2) = \alpha_s(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)) = \frac{\alpha(\mu^2)}{1+\beta_0 \frac{\alpha(\mu^2)}{2\pi} \ln \frac{Q^2}{\mu^2}}$

RENORMALIZATION GROUP INVARIANCE!

THE SLIDING SCALE

$$C_N\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right)A_N(\mu^2)=C_N\left(1,\alpha_s(Q^2)\right)\left[\exp\int_{\mu^2}^{Q^2}\frac{d\lambda^2}{\lambda^2}\gamma_N[\alpha(\lambda^2)]\right]A_N(\mu^2)$$

$$\mu^2\frac{d}{d\mu^2}A_N(\mu^2)=\gamma_N(\mu^2)A_N(\mu^2)\Leftrightarrow\mu^2\frac{d}{d\mu^2}C_N\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right)=-\gamma_N(\mu^2)C_N\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right)$$

THE RUNNING MATRIX ELEMENT!

$$C_N\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right)A_N(\mu^2)=C_N\left(1,\alpha_s(Q^2)\right)A_N(Q^2)$$

RESUMMATION

recall $\beta(\alpha_s)=-\beta_0\alpha_s^2+O(\alpha^3)$; let $\gamma_N(\alpha_s)=\gamma_N^{(0)}\alpha_s+O(\alpha^3)$;

$$\exp\int_{\mu^2}^{Q^2}\frac{d\lambda^2}{\lambda^2}\gamma_N[\alpha(\lambda^2)]=\exp-\int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)}\frac{d\alpha}{\alpha}\frac{\gamma_N^{(0)}}{\beta_0}=\left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{-\frac{\gamma_N^{(0)}}{\beta_0}}$$

$$=\left(1+\beta_0\alpha_s(\mu^2)\ln\frac{Q^2}{\mu^2}\right)^{\frac{\gamma_N^{(0)}}{\beta_0}}+O\left[\alpha^2(\mu^2)\ln\frac{Q^2}{\mu^2}\right]\qquad\text{LEADING LOG}$$

$$=1+\alpha_s(\mu^2)\gamma_N^{(0)}\ln\frac{Q^2}{\mu^2}+O(\alpha^2(\mu^2))\qquad\text{LEADING ORDER}$$

BACK TO THE DIAGRAMS
LO & NLO

- $C_N = \int_0^1 dx x^{N-1} \hat{\sigma}\left(x, \frac{Q^2}{\mu^2}\right)$
- $\hat{\sigma}\left(x, \frac{Q^2}{\mu^2}\right) \Rightarrow$ CROSS SECTION FOR SCATTERING ON FREE QUARK (UP TO TENSOR STRUCT.)
- $C_N = (1 + \alpha_s(Q^2) C_N^{(1)}) \left[1 + \alpha_s(\mu^2)\gamma_N^{(0)} \ln \frac{Q^2}{\mu^2} + O(\alpha^2(\mu^2))\right]$

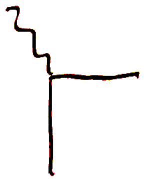
  $\hat{\sigma}(x, \mu^2) = (\delta(1-x) + \alpha_s(Q^2)\hat{\sigma}^{(1)}(x)) \otimes \left(\delta(1-x) + \alpha_s(\mu^2) P^{(0)}(x) \ln \frac{Q^2}{\mu^2}\right) + O(\alpha^2(\mu^2));$

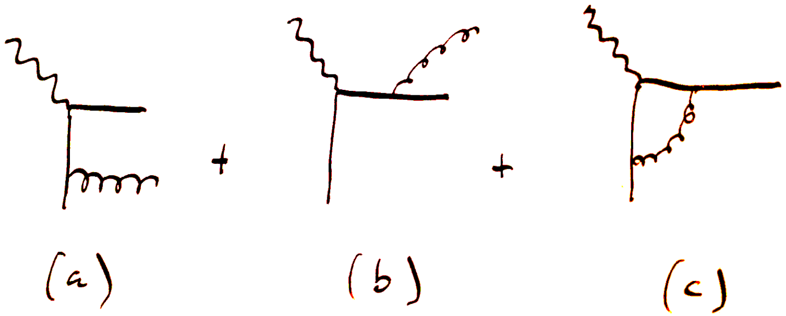
  $\otimes$ = CONVOLUTION INTEGRAL

  $= \delta(1-x) + \alpha_s(\mu^2)\left[\hat{\sigma}^{(1)}(x) + P^{(0)}(x) \ln \frac{Q^2}{\mu^2}\right] + O(\alpha^2(\mu^2))$

- $P^{(0)}(x)$ SPLITTING FUNCTION $\Leftrightarrow \gamma_N^{(0)} = \int_0^1 x^{N-1} P^{(0)}(x)$

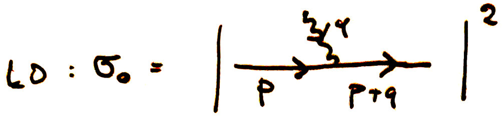
WHERE IS THE LOG?

LEADING ORDER:  $x = x_B \Rightarrow \hat{\sigma}^{(0)}(x) = \delta(1-x);$

NEXT-TO-LEADING ORDER: 

DIAGR. (c) $\propto$ LO=$\delta(1-x) \Rightarrow$ MUST LOOK AT DIAGRAMS (a), (b)

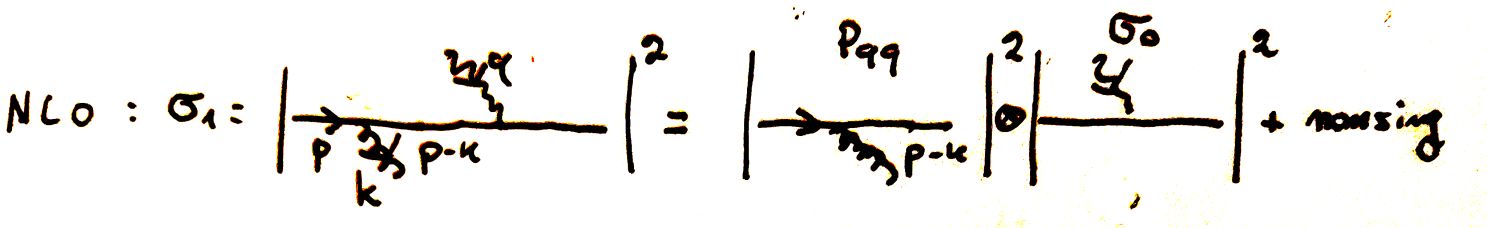
COLLINEAR SINGULARITIES!



LEADING ORDER: $x = x_B \Rightarrow \sigma^{(0)}(x) = \delta(1-x)$;

NLO: $\sigma = \sigma^{(0)}(x) + \frac{1}{2p2q^0(1+v_q)} \int \frac{d^3k}{2E_k(2\pi)^3} |M(pq \to p'k)|^2$

NLO, LEADING LOG:



$M(pq \to p'k) = \frac{\alpha_s}{2\pi} \bar{u}(p')\gamma^\mu \epsilon_\mu(q) \frac{i(\not{p}-\not{k})}{(p-k)^2} \gamma^\nu \epsilon^*_\nu(q) u(p)$

Sudakov parametrization $k = (1-z)p + k_t + \eta$ such that $k^2 = p^2 = \eta^2 = p \cdot k_t = \eta \cdot k_t = 0$

$(p-k)^2 = \frac{-k_t^2}{1-z}$: ON SHELL $\not{p} = \sum_r u^s(p)\bar{u}^s(p) \Rightarrow \frac{i(\not{p}-\not{k})}{(p-k)^2} = \frac{\sum_s u^s(p-k)\bar{u}^s(p-k)}{(p-k)^2} + O(k_t^2)$

$\sigma = \sigma^{(0)}(x) + \frac{1}{2p2q^0(1+v_q)} \int \frac{dk_t^2 dz}{2(1-z)(2\pi)^3} |M(pk \to p-k)|^2 \frac{1}{k_t^2/(1-z)} |M(q(p-k) \to p')|^2$

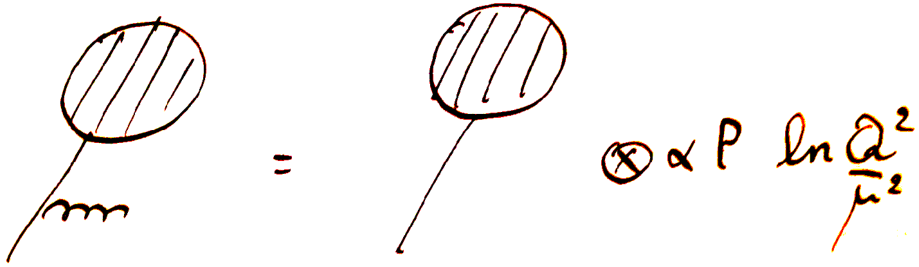
k_t INTEGRAL $\to$ LOG DIVERGENT,

UPPER LIMIT: $\frac{Q^2(1-z)}{4z}$; LOWER LIMIT: μ CUTOFF

$\sigma = \sigma^{(0)}(x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \int dz P(z) \sigma^{(0)}(zp)$

UNIVERSALITY

FORM OF LO $\sigma^{(0)}$ NOT USED IN THE ARGUMENT

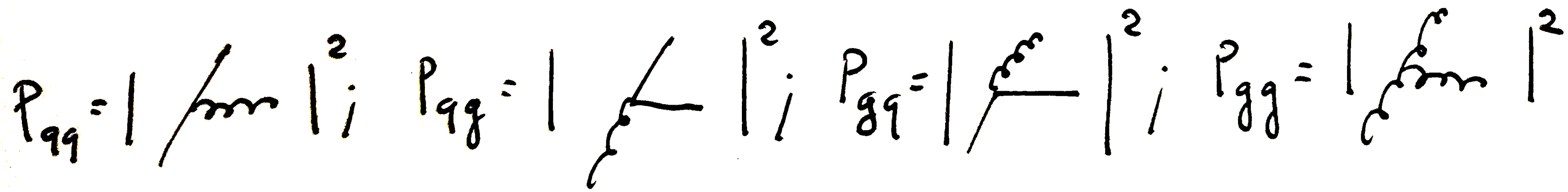


$$\sigma = \sigma^{(0)}(x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \int dz P(z) \sigma^{(0)}(zp) + \text{NON LOGARITHMIC}$$

GENERAL COLLINEAR EMISSION

QUARK CAN RADIATE GLUON, GLUON CAN RADIATE QUARK, GLUON:

$$\sigma_i = \sigma_i^{(0)}(x) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \int dz P_{ij}(z) \sigma_j^{(0)}(zp)$$



FACTORIZATION $\Leftrightarrow$ RESUMMATION

N SPACE

$$C_N\left(\frac{Q^2}{\mu^2},\alpha_s(Q^2)\right)A_N(\mu^2)=C\left(1,\alpha_s(Q^2)\right)\left[\exp\int_{\mu^2}^{Q^2}\frac{d\lambda^2}{\lambda^2}\gamma_N[\alpha(\lambda^2)]\right]A_N(\mu^2)$$

$$=\left(C_N^{(0)}+\alpha_s(Q^2)C_N^{(1)}+\dots\right)\left[1+\alpha_s(\mu^2)\gamma_N^{(0)}\ln\frac{Q^2}{\mu^2}+\dots\right]A_N(\mu^2)$$

$$=\left[C_N^{(0)}+\alpha_s(\mu^2)\left(C_N^{(1)}+\gamma_N^{(0)}\ln\frac{Q^2}{\mu^2}\right)\right]A_N(\mu^2)+O(\alpha_s^2)$$

$$=\left[C_N^{(0)}+\alpha_s(Q^2)C_N^{(1)}\right]A_N(Q^2)+O(\alpha_s^2)$$

THE x SPACE CROSS-SECTION

$$\sigma(x,\alpha_s(Q^2))=\left[\hat\sigma_N^{(0)}+\alpha_s(Q^2)\hat\sigma^{(1)}(x)\right]\otimes q(x,Q^2)$$

$\int_0^1 dx x^{N-1}q(x)=A_N$, $\int_0^1 dx x^{N-1}\hat\sigma=C_N$, $\otimes$ CONVOLUTION INTEGRAL

- $\alpha_s C_N^{(1)}$ IS WHAT IS LEFT OF C AFTER SUBTRACTING THE LOG
- $\alpha_s(Q^2)\hat\sigma^{(1)}(x)$ IS WHAT IS LEFT OF THE NLO XSECT AFTER SUBTR. THE COLLINEAR SING.
- SINGULARITY: UV (UPPER) FOR MATRIX ELEMENT; IR (LOWER) FOR CROSS-SECTION
- THE PDF $q(x,Q^2)$ SATISFIES THE x-SPACE VERSION OF THE RGE (Altarelli-Parisi equation)
- SOLUTION RESUMS COLLINEAR LOGS

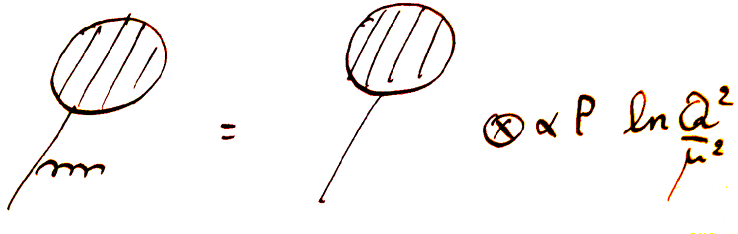
THE ALTARELLI-PARISI EQUATION

$$Q^2 \frac{d}{dQ^2} f_i(x, Q^2) = \sum_j P_{ij}(\alpha_s(Q^2), x) \otimes f_j(Q^2)(x, Q^2)$$

- RESUMS COLLINEAR LOGS IN THE PDFs f_i= QUARKS q_i, ANTIQUARKS $\bar{q}_i$ GLUON g
- COLLINEAR SINGULARITIES SUBTRACTED FROM THE PARTONIC CROSS-SECTION
- COLLINEAR LOGS ARE UNIVERSAL
- THE SCALE DEPENDENCE IS UNIVERSAL
- LOGARITHMICALLY ENHANCED TERMS ALWAYS FACTORIZE

ALTARELLI PARISI: QUARK EMISSION

FACTORIZED COLLINEAR EMISSION, UP TO NON-LOGARITHMIC TERMS:



$$\sigma^{(1)}(\tau, Q^2) = \int_\tau^1 \frac{dx}{x} \sigma^{(0)} \left(\frac{\tau}{x}, Q^2 \right) P(x) \alpha \ln \frac{Q^2}{\mu^2}$$

THE QUARK-QUARK SPLITTING FUNCTION

$$P_{qq}(x) = C_F \left[\frac{1+x^2}{1-x}_+ + \frac{3}{2} \delta(1-x) \right]$$

- THE PARTONIC CROSS-SECTION IS A DISTRIBUTION
- $+$ DISTRIBUTION: $f(x)_+$ ACTS ON $g(x)$ AS $\int_0^1 dx f(x)_+ g(x) \equiv \int_0^1 dx f(x)[g(x) - g(1)]$

DIVERGENCES & LOGS

- COLLINEAR: $\int_{\mu^2}^{(s-Q^2)^2/s} \frac{dk_t^2}{k_t^2} \sim \ln \left[\frac{Q^2}{\mu^2} \frac{(1-\tau)^2}{\tau} \right] \sim \ln \frac{Q^2}{\mu^2}$
- INFRARED (SOFT, LARGE-x, SUDAKOV) $\int_\tau^1 dy \frac{1}{1-y}_+ \sim \ln(1-\tau)$

SOFT EIKONAL EMISSION

$$\bar{u}(p) \qquad \to \qquad \bar{u}(p) ie\gamma^{\mu}(-i)\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\epsilon}$$

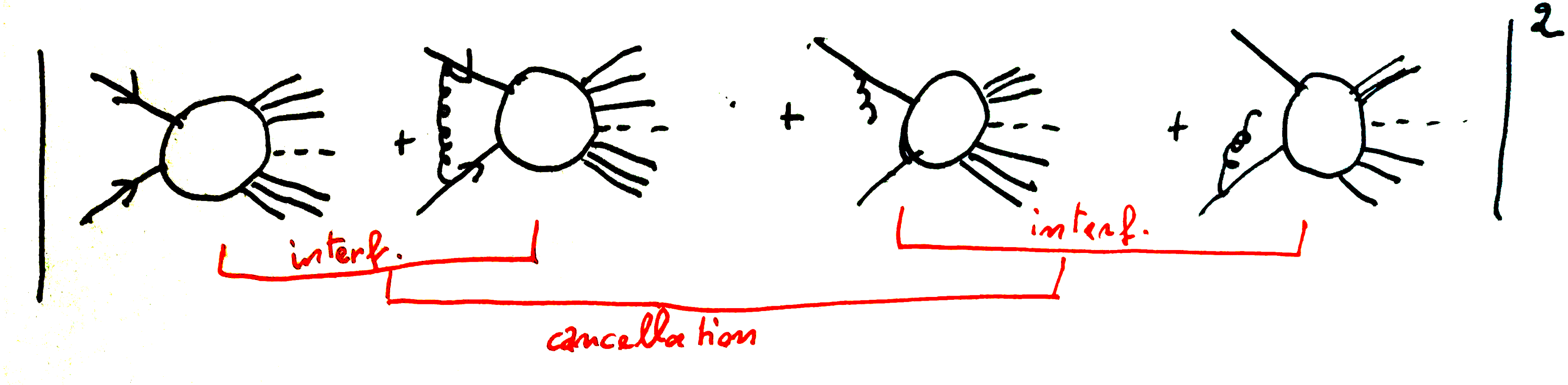
$$= \bar{u}(p)\frac{2p^{\mu}+(m-\not{p})\gamma^{\mu}+O(k)}{2p\cdot k+i\epsilon} = \bar{u}(p)\frac{p^{\mu}}{p\cdot k}$$

SOFT GLUON EMISSION ⇒ UNIVERSAL EIKONAL FACTOR

- EMISSION IS ALSO COLLINEAR
- COLLINEAR EMISSION MAY BE NON-SOFT
- COLLINEAR EMISSION ⇒ UNIVERSAL PARTON-DEPENDENT ALTARELLI-PARISI FACTOR;
- SOFT GLUON EMISSION ⇒ UNIVERSAL EIKONAL FACTOR

CANCELLATION OF IR SINGULARITIES

THE KLN MECHANISM



- CROSS SECTION FOR SINGLE (DOUBLE...) EMISSION INFRARED DIVERGENT:
  $\int dk_z \frac{1}{p \cdot k} = \int \frac{dz}{1-z}$
- DIVERGENCE CANCELLED BY VIRTUAL CORRECTIONS:
  SINGLE EMISSION CANCELLED BY ONE-LOOP,
  DOUBLE EMISSION CANCELLED BY TWO LOOPS ETC:
  REAL $\sim \int \frac{dz}{1-z}$; REAL+VIRTUAL $\sim \int \frac{dz}{1-z}_+$
- AFTER CANCELLATION, LEFTOVER SOFT LOGS
- QED: KLN THEOREM FROM EIKONAL; QCD: ONLY FOR COLOR-SINGLET

SOFT LOGS

x SPACE VS. N SPACE

$$\int_0^1 dx x^{N-1} \frac{\ln^p(1-x)}{(1-x)_+} = \frac{1}{p+1} \ln^{p+1} \frac{1}{N} + O(\ln^p \frac{1}{N})$$

- EACH EMISSION $\to$ EXTRA FACTOR OF $\frac{1}{(1-x)_+} \Leftrightarrow \ln \frac{1}{N}$
- IN x SPACE, EMISSION CONVOLUTIVE $\to$ IN N SPACE, MULTIPLICATIVE
- n-TUPLE EMISSION $\Rightarrow$ EXTRA FACTOR OF $\ln^n \frac{1}{N} \leftrightarrow \frac{\ln^{n-1}(1-x)}{(1-x)_+}$
- NOTE: $x \to 1 \Leftrightarrow N \to \infty$; $O(1-x) \Leftrightarrow O\left(\frac{1}{N}\right)$

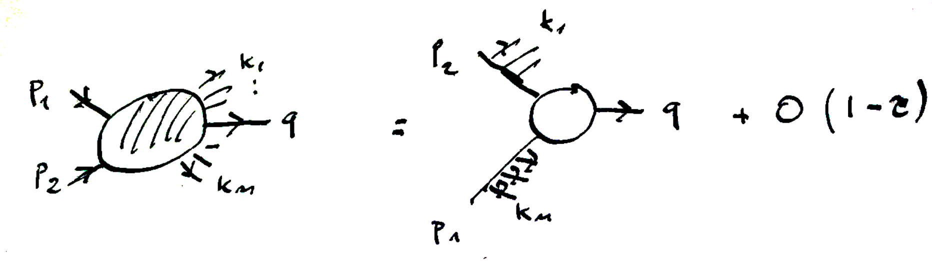
IRC LOGS

- AT EACH PERTURBATIVE ORDER, TWO SOFT LOGS:
  INFRARED $\int_\tau^1 dy \frac{1}{1-y}_+ \sim \ln(1-\tau)$ &
  COLLINEAR $\int_{\mu^2}^{(s-Q^2)^2/s} \frac{dk_t^2}{k_t^2} \sim \ln\left[\frac{Q^2}{\mu^2}\frac{(1-\tau)^2}{\tau}\right] \sim \ln(1-\tau)^2$
- FACTORIZATION SCHEME CHOICE: CAN REDEFINE OPERATOR MATRIX ELEMENT BY ANY FINTE $Z_n(\alpha_s)$: $A_n \to Z_n(\alpha_s)A_n \leftrightarrow$ REDEFINE PDF $q \to Z \otimes q$
- $\overline{\text{MS}}$ (MINIMAL SUBTRACTION) $\Rightarrow$ ONLY $\ln \frac{Q^2}{\mu^2}$ SUBTRACTED FROM $\hat{\sigma}$ INTO PDF EVOLUTION
- $\overline{\text{MS}}$ PARTONIC XSECT CONTAINS TWO EXTRA SOFT LOGS AT EACH EXTRA PERTURBATIVE ORDER

THRESHOLD RESUMMATION & RG INVARIANCE I

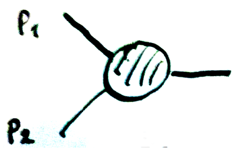
COLORLESS PRODUCTION (HIGGS, DRELL YAN)

FACTORIZATION, KINEMATICS & SCALE SEPARATION



; $q^2 = M^2$; $\tau = \frac{M^2}{s}$; $k^2 \leq \frac{(s^2-M^2)^2}{s}$

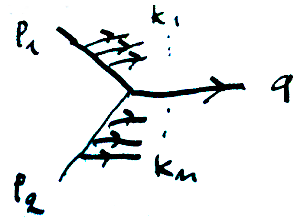
- RADIATION FROM INTERNAL LINES POWER-SUPPRESSED IN SOFT LIMIT
- LOOPS:



$= H(M^2)$

H "HARD" FUNCTION
(LOOP CORRECTIONS TO LEADING-ORDER σ^0)
DIMENSIONLESS, DEPENDS ONLY ON M^2

- REAL EMISSION:



$= J(M^2(1-\tau)^2)$

"JET" FUNCTION(S): DIMENSIONLESS, DEPEND ONLY ON $M^2(1-\tau)^2$ IN SOFT LIMIT:
PHASE-SPACE!

- $\ln \hat{\sigma}(M^2, \tau) = \ln H(M^2) + \ln J(M^2(1-\tau)^2)$ PARTONIC XSEC FACTORIZES INTO $f(M^2)$ (HARD SCALE) & $f(M^2(1-\tau)^2)$ (SOFT SCALE)

THRESHOLD RESUMMATION & RG INVARIANCE II

COLORLESS PRODUCTION (HIGGS, DRELL YAN)

RG IMPROVEMENT

- MELLIN TRANSFORM $F(M^2(1-\tau)^2) \Leftrightarrow F\left(\frac{M^2}{N^2}\right)$ FACTORIZES PHASE SPACE
- MELLIN-SPACE PARTONIC CROSS-SECTION:
  $\ln \hat{\sigma}\left(M^2, \mu^2, N, \alpha(\mu^2)\right) = \ln H\left(\frac{M^2}{\mu^2}, \alpha(\mu^2)\right) + \ln J\left(\frac{M^2/N^2}{\mu^2}, \alpha(\mu^2)\right)$
- $\hat{\sigma}$ IS NOT RG INVARIANT (NOT PHYSICAL OBSERVABLE); $\gamma^{\rm phys} = M^2 \frac{d}{dM^2}\hat{\sigma}$ IS RG INVARIANT
  MELLIN SPACE $\hat{\sigma}$ MULTIPLICATIVELY RENORMALIZED:
  $\sigma^{\rm ren}(N, M^2, \alpha^r(\mu^2)) = Z(N, \alpha^r(\mu^2))\sigma^0(N, M^2, \alpha^0(\mu^2))$
- DEFINE PHYSICAL ANOMALOUS DIMENSION $\gamma^{\rm phys} = M^2 \frac{d}{dM^2}(\ln H + \ln J) = \gamma^c + \gamma^l$
  $\gamma^c = M^2 \frac{d}{dM^2} \ln H\left(\frac{M^2}{\mu^2}, \alpha(\mu^2)\right); \qquad \gamma^l = M^2 \frac{d}{dM^2} \ln J\left(\frac{M^2/N^2}{\mu^2}, \alpha(\mu^2)\right)$
- RG INVARIANCE CONDTION $\mu^2 \frac{d}{d\mu^2}\gamma^{\rm phys} = 0$ BUT γ^l, γ^c NOT SEPARATELY RGI
- $\Rightarrow \mu^2 \frac{d}{d\mu^2}\gamma^l\left(\frac{M^2/N^2}{\mu^2}, \alpha(\mu^2)\right) = -\mu^2 \frac{d}{d\mu^2}\gamma^c\left(\frac{M^2}{\mu^2}, \alpha(\mu^2)\right) = g(\alpha(\mu^2))$
  LOOKS LIKE A RG EQUATION!
- SOL: $\gamma^{\rm phys}\left(N, \frac{M^2}{\mu^2}, \alpha(\mu^2)\right) = \bar{g}_0\left(\alpha(M^2)\right) + \int_{M^2}^{M^2/N^2} \frac{d\mu^2}{\mu^2} g\left[\alpha(\mu^2)\right]$

THE STRUCTURE OF RESUMMED TOTAL CROSS-SECTIONS I

PARTONIC CROSS SECTION IN THE SOFT LIMIT

$$\hat{\sigma}\left(N, \frac{M^2}{\mu_F^2}, \alpha(\mu^2)\right) = H(\alpha(M^2)) \exp \int_{\mu_F^2}^{M^2} \frac{d\mu^2}{\mu^2} \int_1^{N^2} \frac{dn}{n} g\left[\alpha(\mu^2/n)\right] = C_{\rm res}(N, \alpha_s)$$

RESUMMATION AS RG EVOLUTION DOWN TO SOFT SCALE WITH
"SOFT AN. DIM." $g[\alpha] = c_g^1 \alpha + c_g^2 \alpha^2 + \dots$

LOG COUNTING

$$C_{\rm res}(N, \alpha_s) = g_0(\alpha_s) \exp\left[\ln N g_1(\alpha_s \ln N) + g_2(\alpha_s \ln N) + \alpha_s g_3(\alpha_s \ln N) + \dots\right];$$
$$g_0(\alpha_s) = 1 + \alpha_s g_{0,1} + \alpha_s^2 g_{0,2} + O(\alpha_s^3); \quad g_1(\lambda) = \textstyle\sum_{k=2}^{\infty} g_{1,k}\lambda^k, \; g_i(\lambda) = \sum_{k=1}^{\infty} g_{i,k}\lambda^k \text{ FOR } i \geq 2$$

LOG APPROX.	XSECT ACCURACY	EXP. ACCURACY: $\alpha_s^n L^k$	g_0 ACCURACY: α_s^i
LL	$k = 2n$	$k = n + 1$	0
NLL	$2n - 2 \leq k \leq 2n$	$k = n$	1
NNLL	$2n - 4 \leq k \leq 2n$	$k = n - 1$	2

NOTE ACCURACY OF g_0 ONE ORDER HIGHER THAN CORRESP. LOG ACCURACY $\Rightarrow$ INCREASES LOG ACCURACY OF $\hat{\sigma}$ BY ONE ORDER

THE STRUCTURE OF RESUMMED TOTAL CROSS-SECTIONS II

RG INVARIANT EXPRESSION

$$C_{\rm res}(N,\alpha_s) = \bar{g}_0(\alpha_s)\exp\left[\int_1^{N^2}\frac{dn}{n}\int_{M^2 n}^{M^2}\frac{d\mu^2}{\mu^2}\,g\left[\alpha(\mu^2/n)\right]\right]$$
$$= \hat{g}_0(\alpha_s)\exp\left[2\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\int_{M^2}^{M^2(1-x)^2}\frac{d\mu^2}{\mu^2}\,\hat{g}\left[\alpha(\mu^2)\right]\right]$$

- TO BE USED WITH PDFS EVALUATED AT $\mu_f^2 = M^2$
- $\hat{g}$ DETERMINED ORDER BY ORDER BY g
- $\hat{g}$ DETERMINED BY MATCHING TO FIXED ORDER

CUSP ANOMALOUS DIMENSION

$$C_{\rm res}(N,\alpha_s) = \hat{g}_0(\alpha_s)\exp\left[2\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\int_{M^2}^{(1-z)^2 M^2}\frac{dq^2}{q^2}A_g^{\rm th}\left(\alpha_s\left(q^2\right)\right) + D_g^{\rm th}\left(\alpha_s\left((1-z)^2 M^2\right)\right)\right]$$

- A AND D POWER SERIES IN α_s
- LEADING LOG $\Rightarrow$ LEADING ORDER $A \leftrightarrow$ COEFFICIENT OF $\frac{1}{1-x}_+$ IN SPLITTING FUNCTION
- $O(\alpha_s^n)$ CONTRIBUTION TO A DEFINED AS $O(\alpha_s^n)$ COEFFICIENT OF $\frac{1}{1-x}_+$ IN SPLITTING FUNCTION: CUSP ANOMALOUS DIMENSION
- D STARTS AT NNLO, DUE TO LARGE-ANGLE GLUON EMISSION
- IF FINAL STATE CAN RADIATE (E.G. DIS), FURTHER D-LIKE "B" TERM DUE TO FINAL-STATE COLLINEAR RADIATION

SUMMARY

- OPERATOR MATRIX ELEMENTS DIVERGENT $\Rightarrow$ UV RENORMALIZATION
- RG INVARIANCE OF PHYSICAL OBSERVABLE $\Rightarrow$ PHYSICAL SCALE LOG RESUMMATION
- OPERATOR UV LOGS $\Leftrightarrow$ COEFFICIENT COLLINEAR LOGS
- RESUMMATION $\Leftrightarrow$ FACTORIZATION
- COLLINEAR UNIVERSALITY VS SOFT (EIKONAL) UNIVERSALITY
- SCALE SEPARATION + RG INVARIANCE $\Rightarrow$ RESUMMATION OF ANOMALOUS DIMENSION