UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE

Corso di Laurea Magistrale in Fisica

Resummation of rapidity distributions in the singly and doubly soft limits

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Introduction

Years of efforts in trying to describe fundamental physics produced the so-called Standard Model or rather the theory of fundamental interaction. This is a theory which provides a picture of 3 of the 4 fundamental forces (electromagnetic, weak and strong interactions). In order to validated the theory, many experiments have been conducted, exploring its effectiveness in describing particle physics at increasingly higher energies. In order to provide the necessary data to test the theory, particle accelerators have been built, where different types of particles (leptons or hadrons) are accelerated and are made to collide between each other and then the particle outcome is measured. The simplest observable which can be both computed from the theoretical model and measured by the experiments is the so called *inclusive cross section*. It represents the rate of a certain type of events, once the colliding particles are fixed.

The theory has passed many experimental tests. Among them, probably the most popular one is the prediction of the existence of the Higgs boson, but there are still some physical ingredients which are missing. Two examples are dark matter, which is believed to be composed of particles still not included, and gravity, which we are not able to represent as a fundamental force within the framework of quantum field theory. In order to obtain hints on how to add these and other phenomena to the theory, we need to get increasingly stringent constraints from experiments. This can be done by raising the energy of the particle collisions, or by measuring more specific observables, such as *differential cross sections*, namely cross sections where some kinematic characteristics of the final state has been fixed.

This thesis points in the latter direction. In fact, we will focus on the production of a massive final state (a Higgs boson, a Z boson or a virtual photon), with fixed longitudinal rapidity (along the collision axe). Inclusive and differential cross sections can be computed perturbatively, namely as an infinite sum of terms with decreasing value, therefore computing a finite number of them should provide an approximation of their true value. The condition that the terms of the series become less and less important is not always correct, in particular it is known to fail near the kinematic thresholds, for instance when the collision energy is barely sufficient to produce a Higgs boson, or when the Higgs boson has the maximum longitudinal momentum.

In these cases, the perturbative approximation is spoiled because some logarithmic terms become of the same order of magnitude at any perturbative order. The solution to this problem is to reorganise the perturbative series and to collect all the terms which are of the same magnitude from every perturbative order. The reorganised series can be obtained by the so called *resummation formulas*, which

can be derived following different approaches. In this thesis we aim to obtain the resummation formulas for the rapidity distribution of the Higgs production in the two mentioned threshold behaviour: doubly soft limit (barely sufficient energy) and singly soft limit (maximum longitudinal momentum).

In chapter 1 we provide a brief description of the strong interaction sector of the fundamental interaction theory, known as Quantum Chromodynamics. We recall the QCD Lagrangian and its fundamental properties. Among them, asymptotic freedom and colour confinement are particularly important. Then, we briefly describe the fixed order computation notation. Finally, we explain the emergence of IR divergence from the emission of soft or collinear massless particle emission, and their cancellation between real and virtual contribution.

In chapter 2, we face a problem due to colour confinement. In fact, since single quark states can not be observed in nature, the collisions must take place between hadrons, which are composite particles, and therefore are not fundamental objects of the theory. Moreover, we are not able to provide either an exact treatment of the hadrons or a perturbative one. The solution to this problem is the so-called factorization, and in this chapter we describe a formal derivation for the case of deep inelastic scattering. From this derivation parton distribution functions emerge, and we describe some of their key features. Finally, we report the most commonly used kinematic variables for deep inelastic scattering and hadron collision.

In chapter 3, we motivate the emergence of logarithmic enhanced terms and we characterise them in different classes, some of which we aim to resum. Moreover, we describe how these big logs are mapped into big logs by the Mellin transform. Then, we describe the renormalization group argument which allows us to derive the resummation formulas. Firstly, we apply it to the case of inclusive cross sections, and then we generalise it for differential cross sections, which will be called multi-scale case.

Chapter 4 represents the core of the thesis. Firstly, we describe the notation used to refer to physical quantities, in order to later highlight the threshold behaviours. Then, in order to obtain the factorization of the rapidity distribution, we describe the necessity of a double transform (Mellin-Fourier or Mellin-Mellin). After providing a brief description of two examples of rapidity distributions in their most common forms, we derive to which variable limits the thresholds correspond, both in Mellin-Fourier and Mellin-Mellin space. Then, thanks to a study of the phase space structure, we derive the soft and collinear scales of our process, both in the singly and doubly soft kinematic limit. Once we obtained them, we are ready to substitute them into the resummation formulas. We prove that in the doubly soft case, the contributions from the soft scale are only subleading, while in the singly soft limit both soft and collinear scales are relevant. We also prove that in the doubly soft limit the resummed expression is identical to the one of the inclusive cross section up to an algebraic substitution.

Chapter 1

The strong interaction

In this chapter we summarize the key theoretical ingredients of Quantum Chromodynamics. Our aim is to recall the basic features of QCD, and in particular those which will be important in chapters 3 and 4, which are the core of this work. In particular, we briefly describe the QCD Lagrangian, we recall the properties of asymptotic freedom and colour confinement, the perturbative technique and its notation. Finally, we provide some examples of the treatment of IR divergences, their factorization and cancellation properties.

1.1 Quantum Chromodynamics

Quantum Chromodynamics is the theory which describes the strong interaction. It is a gauge theory with SU(N_C) gauge group, where $N_C = 3$. It takes place between 1/2-spin fermions, called *quarks*, which are represented by Dirac spinors ψ. There are six known types of quarks, which are identical with respect to the strong interaction, but they have different masses (among other differences). In order to highlight their common strong behaviour, it is usually said that quarks come in six different *flavours*, and they are denoted by ψ_f, with $f = 1, ..., N_f$. The strong interaction between quarks is mediated by 1-spin massless bosons, called *gluons*, which are the gauge bosons of the theory and are denoted by A_μ. Quarks live in the fundamental representation of SU(3), therefore they are represented by ψ_f^i, where $i = 1, 2, 3$ (3 is the dimension of the representation space), while gluons live in the adjoint representation, hence they are represented by A_μ^a, $a = 1, ..., 8$.

The QCD Lagrangian reads as follows:

$$\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f^i (i\not{D} - m_f)^{ij} \psi_f^j - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}, \tag{1.1}$$

where the sum is over the flavours f,

$$F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^b \tag{1.2}$$

is the field strength tensor of the gluon field A_μ^a, f^{abc} are the structure constant of the SU(3) algebra and g_s is the coupling constant. In addition to g_s, it is often

Flavours	Up (u)	Down (d)	Charm (c)	Strange (s)	Top (t)	Bottom (b)
Masses	2.5 MeV	5.0 MeV	0.1 GeV	1.3 Gev	4.2 GeV	173 GeV

Table 1.1: The six different flavours of quarks, with their masses and their electric charges

convenient to use $\alpha_s := \frac{g_s^2}{4\pi}$. The different quark masses m_f can be found in table 1.1. The covariant derivative D_μ is defined as

$$D_\mu^{ij} := \mathbb{I}^{ij}\partial_\mu - ig_s A_\mu^a T^{aij}, \tag{1.3}$$

where T_a^{ij} are the generators of the SU(3) algebra in the fundamental representation. They are often chosen to be $T_a^{ij} = \lambda_a^{ij}/2$, where λ_a are the Gell-Mann matrices. The Lagrangian (1.1) is invariant under the following gauge transformations:

$$\begin{cases} \psi(x) \longmapsto U(x)\psi(x) \\ A_\mu(x) \longmapsto U(x)A_\mu U^\dagger(x) + \frac{i}{g_s}U(x)\partial_\mu U^\dagger(x), \end{cases} \tag{1.4}$$

where $U(x) = \exp\{i\theta^a(x)T^a\} \in SU(3)$.

1.2 Gauge fixing and ghosts

For non abelian gauge theories in general and for QCD in particular, it is well-known that the usual quantization procedure, i.e. naively imposing the canonical quantization relations, does not work. This is due to the gauge invariance of such theories, in fact we are considering an infinite number of gauge equivalent field configurations, and we need to constrain the gauge field configurations space. This can be done in the path-integral formulation of Quantum Field Theory by means of Faddeev-Popov formula (see [2] and [6]). It can be shown that this procedure is equivalent to adding to the Lagrangian (1.1) a gauge fixing term and a ghost term. Then, the QCD Lagrangian reads as follows:

$$\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{fix} + \mathcal{L}_{ghost}, \tag{1.5}$$

where

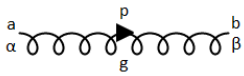
$$\mathcal{L}_{fix} := -\frac{f(A)^2}{2\xi} \qquad \text{and} \qquad \mathcal{L}_{ghost} := \bar{c}\partial^\mu D_\mu c. \tag{1.6}$$

In definitions (1.6), $f(A)$ is the function chosen for the gauge field configuration contraint $f(A) = 0$, ξ is an arbitrary parameter and $c(x)$ are gauge-dependent fermionic spinless fields, known as *ghosts*.

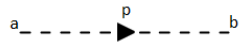
We recall the fact that no physical quantity can depend on the choice of $f(A)$ and ξ, because the theory is invariant under gauge transformations, therefore fixing a gauge can not affect the theoretical predictions of the theory. Moreover, it can be shown that the ghosts fields we introduced are not physical particles, but they play the role of negative degrees of freedom, namely they cancel the contributions from the longitudinal and time-like polarization of the gluons. A systematic treatment of gauge fixing and a precise explanation of the physical interpretation of the ghost fields can be found in [2].

1.3 Feynman rules of QCD

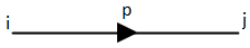
In the Quantum Field Theory framework, transition amplitudes are computed by summing over all possible Feynman diagrams allowed by the Feynman rules of the considered theory. Here we report the Feynman rules of the Lagrangian (1.5) in the Feynman gauge, i.e. $\partial_\mu A^\mu = 0$ and $\xi = 1$:



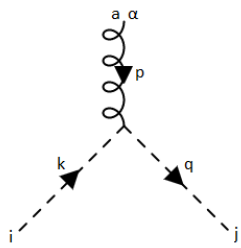
$$= \delta^{ab} \frac{-ig^{\alpha\beta}}{p^2 + i\epsilon} \tag{1.7}$$



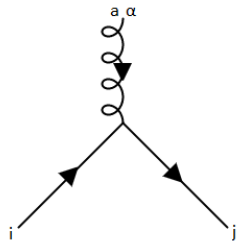
$$= \delta^{ab} \frac{i}{p^2 + i\epsilon} \tag{1.8}$$



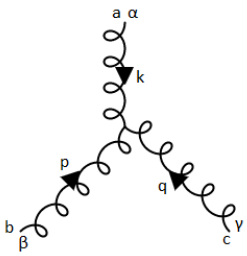
$$= \delta^{ji} \frac{i}{\not{p} - m + i\epsilon} \tag{1.9}$$



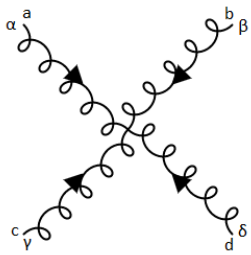
$$= -g_s f^{abc} q^\alpha \tag{1.10}$$



$$= i g_s T^a_{ji} \gamma^\alpha \tag{1.11}$$



$$= \begin{aligned} & g_s f^{abc} [g^{\alpha\beta}(k-p)^\gamma + \\ & g^{\beta\gamma}(p-q)^\alpha + g^{\gamma\alpha}(q-k)^\beta] \end{aligned} \tag{1.12}$$



$$= \begin{aligned} & -ig_s^2 f^{eac} f^{ebd} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}) \\ & -ig_s^2 f^{ead} f^{ebc} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}) \\ & -ig_s^2 f^{eab} f^{ecd} (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \end{aligned} \tag{1.13}$$

1.4 Running coupling and asymptotic freedom

Amplitude computations involve loop integrals, (as in eq. (1.26)), which may present UV divergences. The well-known procedure of renormalization removes divergences and makes the coupling constant acquire a dependence $\alpha_s = \alpha_s(\mu_R^2)$ on the renormalization scale μ_R^2. Its dependence can be described by defining the *beta function*:

$$\beta(\alpha_s(\mu_R^2)) := \frac{d \ln \alpha_s(\mu_R^2)}{d \ln \mu_R^2}. \tag{1.14}$$

The beta function can be computed as a power series in the coupling constant as follows:

$$\beta(\alpha_s(\mu^2)) = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 + O(\alpha_s^4), \tag{1.15}$$

where, for a generic $\mathrm{SU}(N_C)$ gauge theory with N_f flavours, the first coefficient is

$$\beta_0 = \frac{11 N_C - 2 N_f}{12\pi}. \tag{1.16}$$

The solution of equation (1.14) at first non trivial order is:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_R^2)}{1 + \beta_0 \alpha_s(\mu_R^2) \ln \frac{Q^2}{\mu_R^2}}. \tag{1.17}$$

For $N_C = 3$ and $N_f \leq 16$, we have $\beta_0 > 0$. Therefore, for the case of QCD, α_s decreases as μ_R^2 increases, and the theory is said to be *asymptotically free*. On the other hand, as μ^2 decreases, α_s increases, and it hits the *Landau pole* at

$$Q = \Lambda_{QCD} = \mu_R \exp\left\{ -\frac{1}{2\beta_0 \alpha_s(\mu_R^2)} \right\}. \tag{1.18}$$

Actually, the Landau pole is not a consistency problem of the theory because, before reaching $Q = \Lambda_{QCD}$, the coupling constant has already reached the value $\alpha_s \approx 1$, and therefore the perturbative approximation (1.15) is no longer valid, hence the rest of the above derivation.

1.5 Callan-Symanzik equation

In section 1.4 we stated that, after the renormalization procedure, the coupling constant acquires a dependence on the renormalization scale $\alpha_s = \alpha_s(\mu_R^2)$. Moreover, the generic transition amplitude $M(Q^2, \alpha_s)$ under consideration acquires a dependence on μ_R^2, too, i.e. $M(Q^2, \mu_R^2, \alpha_s(\mu_R^2))$. We recall the fact that the renormalization scale μ_R^2 is an arbitrary scale, and therefore no physical observables can depend on it. For a generic observable O, this statement is equivalent to the Callan-Symanzik equation:

$$\frac{dO(Q^2, \mu_R^2, \alpha_s(\mu_R^2))}{d \ln \mu_R^2} = 0. \tag{1.19}$$

By computing the derivative of a composite function, the equation can be written in the usual form:

$$\frac{\partial O(Q^2, \mu_R^2, \alpha_s)}{\partial \ln \mu_R^2} + \beta(\alpha_s(\mu_R^2)) \frac{\partial O(Q^2, \mu_R^2, \alpha_s)}{\partial \alpha_s} = 0, \tag{1.20}$$

where we recognised the definition of the β function (1.14). From this equation, we can read that the explicit dependence of O on μ_R^2 exactly compensates the implicit dependence of O through $\alpha_s(\mu_R^2)$.

1.6 Colour confinement

Colour confinement is the property of QCD that, below approximately 150 MeV, colour charged particles can not be directly observed. Usually, we refer to the Landau pole Λ_{QCD} as an indication for the threshold of confinement. Below this scale, only colourless states can be directly observed. These states are called *hadrons* and are bound states of the fundamental particles of the theory (quarks and gluons). Hadrons are divide in two classes: barions, composed of three quarks, and mesons, which are a quark-antiquark pair. This physical phenomenon has been directly observed, but, to date, there is not a proof of QCD confinement because, at this energy scale the value of the coupling constant is $\alpha_s \gtrsim 1$, hence the perturbative techniques fail. However, lattice QCD simulations show the presence of a phase transition compatible with the observed phenomenon.

1.7 Perturbative series

In the Quantum Field Theory framework, the main tool to compute a desired observable O is expanding it in a series in coupling constant powers, i.e.

$$O = \sum_{n=n_0}^{\infty} O_n \alpha_s^n. \tag{1.21}$$

In the above expression the first non trivial contribution, i.e. the term proportional to $\alpha_s^{n_0}$ is called *leading order* (LO), the second non trivial term is called *next-to-leading order* (NLO) and the k-th non trivial term is denoted with N^kLO. The partial sums of the above series are increasingly better approximations of O if $\alpha_s \ll 1$. With the expression "*fixed order* computation", we refer to the approximation of the series with its partial sum up to N^kLO, for some chosen k, i.e.

$$O_k^{fix} = \sum_{n=n_0}^{k} O_n \alpha_s^n. \tag{1.22}$$

1.8 IR divergences

In addition to UV divergences, in QCD amplitude computations also IR divergences arise because of the presence of gluons, which are massless particles. With the term *infrared*, we refer to two different types of kinematic limit:

- soft particles, which means that all the components of its 4-momentum are approaching 0;
- collinear particles, which means that its 4-momentum is becoming collinear to the 4-momentum of the particle which emitted it.

1.8.1 Soft singularities

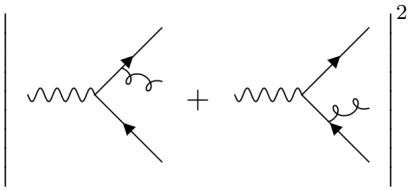
In this section, we provide a specific example to describe the factorization of soft emissions, the emergence of soft singularities and their cancellation properties. We consider a virtual photon decay into a quark-antiquark pair in the high energy limit, where the quark masses can be neglected. The NLO contributions to the cross section can be computed as:

$$\sigma^{NLO} = \sigma^{NLO}_{real} + \sigma^{NLO}_{virt}, \tag{1.23}$$

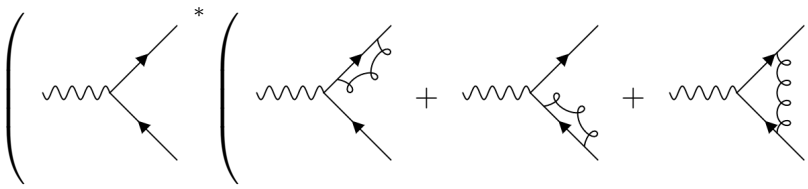
where

$$\sigma^{NLO}_{real} = \frac{1}{2E^2_{cm}} \int d\phi_3 \, |M_{real}|^2 \qquad \text{and} \qquad \sigma_{virt} = \frac{1}{2E^2_{cm}} \int d\phi_2 \, |M_{virt}|^2, \tag{1.24}$$

and $d\phi_n$ is the n-particle phase space integration measure. The squared modulus of the amplitudes are obtained as follows:

$|M_{real}|^2 =$  (1.25)

and

$|M_{virt}|^2 = 2\,\mathrm{Re}\Bigg($  $\Bigg)$. (1.26)

If we write the real emission contribution to the amplitude (1.25), we get:

$$\begin{aligned} M^{\mu}_{real}(q; p_1, p_2, k) = \bar{u}(p_1)\Bigg(& (-ig_s T^a \gamma^\nu) \frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2} (-ie\gamma^\mu) \\ & + (-ie\gamma^\mu) \frac{i(-\not{p}_2 - \not{k})}{(p_2+k)^2} (-ig_s T^a \gamma^\nu) \Bigg) v(p_2) \epsilon^a_\nu(k), \end{aligned} \tag{1.27}$$

where p_1 and p_2 are respectively the 4-momenta of the final state quark and anti-quark, k is the 4-momentum of the gluon and $\epsilon^a_\mu(k)$ its polarization vector. In the soft gluon limit of this amplitude, i.e. $k^\mu \to 0$, M^μ_{real} becomes:

$$\begin{aligned} & M^{\mu}_{real}(q; p_1, p_2, k) = \\ & \quad = g_s \bar{u}(p_1) \left(T^a \gamma^\nu \frac{\not{p}_1}{2(p_1,k)} (-ie\gamma^\mu) - (-ie\gamma^\mu) \frac{\not{p}_2}{2(p_2,k)} T^a \gamma^\nu \right) v(p_2) \epsilon^a_\nu(k). \end{aligned} \tag{1.28}$$

By exploiting the relation $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$ to switch the order of $\not{p}_1$ and $\not{p}_2$ with γ^ν, and by exploiting the Dirac equations of motion for a massless quark $\bar{u}(p)\not{p} = 0$ and $\not{p}v(p) = 0$, we obtain:

$$M^\mu_{real}(q; p_1, p_2, k) = \left(\epsilon^a(k), \frac{p_1}{(p_1, k)} - \frac{p_2}{(p_2, k)}\right) \bar{u}(p_1)(-ie\gamma^\mu T^a)v(p_2). \tag{1.29}$$

By computing the unpolarized squared modulus of the amplitude and by exploiting the relation

$$\sum_{spins} \epsilon_\mu(k)\epsilon_\nu(k) \left(\frac{p_1^\mu}{(p_1, k)} - \frac{p_2^\mu}{(p_2, k)}\right) \left(\frac{p_1^\nu}{(p_1, k)} - \frac{p_2^\nu}{(p_2, k)}\right) = \frac{2(p_1, p_2)}{(p_1, k)(p_2, k)}, \tag{1.30}$$

we finally get:

$$|M_{real}(q; p_1, p_2, k)|^2 = g_s^2 C_F \frac{(p_1, p_2)}{(p_1, k)(p_2, k)} |M(q; p_1, p_2)|^2 \tag{1.31}$$

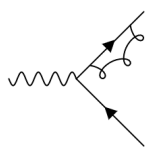
where $M(q; p_1, p_2)$ is the LO order contribution to the amplitude. If we substitute this expression in equation (1.24), we obtain:

$$\sigma^{NLO}_{real} = \frac{C_F g_s^2}{2E_{cm}} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6 4E_1E_2} |M(q; p_1, p_2)|^2 (p_1, p_2) \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{1}{(p_1, k)(p_2, k)}, \tag{1.32}$$

which is a divergent integral in the region of $k^\mu \to 0$. This is an example of a soft singularity.

By looking at expression (1.31), we recognise that the real gluon emission corrections to the LO factored in a product of the LO and the factor $\frac{(p_1,p_2)}{(p_1,k)(p_2,k)}$ in the soft gluon limit. These are called *Eikonal factors*, and the factorization (1.31) is a general property of soft emission, and does not depend on the considered process.

Now we consider the contributions to σ^{NLO}_{virt} and in particular we consider the following amplitude:



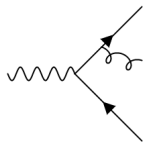
$$= \bar{u}(p_1)(-ig_s T^a \gamma^\nu) \left[\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{i(\not{p}_1 - \not{k})}{(p_1 - k)^2}\right] (-ig_s T^a \gamma_\nu)(-ie\gamma^\mu)v(p_2). \tag{1.33}$$

As in the real gluon emission case, also the virtual contributions are singular. In fact, as we can see from expression (1.33), this is due to the fact that the loop integral is divergent in the soft gluon region.

In [4] there is a proof that all soft divergences from loop integrals are cancelled by soft divergences coming from phase space integrals of real emission contributions. Therefore, in the example above, once σ^{NLO}_{real} and σ^{NLO}_{virt} have been regularized, by adding them together we obtain a finite contribution. The full computation of the virtual photon decay can be found in [8], where the cancellation of soft divergences is also checked.

1.8.2 Collinear singularities

In this section, in order to analyse the emergence of collinear singularities and their cancellation properties, we consider again the virtual photon decay of section 1.8.1. We examine the following real gluon emission amplitude:



$$= \bar{u}(p_1)(-ig_s T^a \gamma^\nu)\frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2)\epsilon^a_\nu(k). \tag{1.34}$$

We substitute the identity $\sum_{spins} u(p+k)\bar{u}(p+k) = \not{p} + \not{k}$ and we obtain:

$$= \frac{1}{2(p_1,k)}\sum_{spins}\left[\bar{u}(p_1)(g_s T^a\gamma^\nu)u(p_1+k)\epsilon^a_\nu(k)\right]\left[\bar{u}(p_1+k)(-ie\gamma^\mu)v(p_2)\right]$$

$$= \frac{1}{2(p_1,k)}\sum_{spins} M^\mu(q;p_1+k,p_2)M(p_1+k;p_1,k). \tag{1.35}$$

It is convenient to introduce the following approximated parametrisation of the 4-momenta p_1 and k in the collinear limit:

$$p_1 = zp + k_T - \frac{\eta k_T^2}{2z(p_1,\eta)} \qquad k = (1-z)p - k_T - \frac{\eta k_T^2}{2(1-z)(p_1,\eta)}, \tag{1.36}$$

and $p_1^2 = k^2 = (p,k_T) = (\eta,k_T) = 0$, hence $p^2 = (p_1+k)^2 = 2(p_1,k) = -\frac{k_T^2}{z(1-z)}$. If we take the square modulus of the amplitude and we compute

$$\frac{1}{2}\sum_{spins}|M(p_1+k;p_1,k)|^2 = 2g_s^2 C_F k_T^2 \frac{1+z^2}{1-z}, \tag{1.37}$$

the contribution $\tilde{\sigma}$ of this particular squared amplitude to the cross section is:

$$\tilde{\sigma} = \frac{\alpha_s}{4\pi}\int dz \int \frac{dk_T^2}{k_T^2} C_F \frac{1+z^2}{1-z}\sigma^{LO}(p). \tag{1.38}$$

We can immediately recognise that the dk_T^2 integral is logarithmically divergent, which is an example of a collinear singularity.

As in the case of soft singularities, some divergences are removed, but not all of them. In fact, in [4] there is also a proof that, once we add real and virtual emission corrections, the result is free from collinear singularities due to gluons emitted from *final* state particles, while collinear singularities from initial state particles are not removed. Therefore, the example discussed in this section is free of collinear divergences, because the only initial state particle is a virtual photon, which can not radiate a collinear gluon, hence the collinear gluons must be radiated from the final state quark and antiquark.

Differently, if we consider a deep inelastic scattering process (DIS), i.e.

$$\gamma^* + q \longrightarrow q + X, \tag{1.39}$$

the process has an incoming quark. If we compute again the cross section including both real and virtual corrections, we obtain:

$$\sigma^{NLO} = \frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{k_T^2} \int dz\, P_{q\to q}(z) \sigma^{LO}(zp), \tag{1.40}$$

where

$$P_{q\to q}(z) \coloneqq C_F \frac{1+z^2}{[1-z]_+}. \tag{1.41}$$

$P_{q\to q}(z)$ is called quark-to-quark *splitting function*. It is a plus distribution (see A.2) and it does not depend on the considered process, but only on the species of quarks/gluons involved in the splitting amplitude and on the variable z.

We can see that the soft singularity $\frac{1}{1-z}$ in the limit $z \to 1$ is cancelled in the integration with the splitting function because of the numerator $\sigma^{LO}(zp) - \sigma^{LO}(p)$, which vanishes in this limit. On the other hand, the collinear singularity is still present. The solution to the problem of initial state collinear singularities will be provided in chapter 2.

Chapter 2

Factorization

In section 1.6, we discussed the phenomenon of colour confinement, which forbids to observe free quark states. Therefore, the initial state particles are hadrons, which are not fundamental 1-particle states of the theory. To overcome the problem, we exploit the QCD property of factorization. In this chapter we provide a proof of factorization for DIS and afterwards a discussion of the solution to the initial state collinear singularities problem within this framework. Then, we outline the PDFs properties and we describe the application of the factorization to DIS and hadron collision.

2.1 Factorization for DIS

In this section we provide a formal derivation of factorization for a DIS process between a hadron and a hard ($Q^2 = -q^2 \to \infty$) virtual photon, i.e.

$$h(P) + \gamma^*(q) \longrightarrow X(P_X). \tag{2.1}$$

For a matter of simplicity, we drop parton indices (quark flavours and gluons), which will be restored at the end of this argument. We start by computing the total cross section σ_{tot} for this process thanks to the optical theorem, which reads:

$$\sigma_{tot} = \int d\phi_X \, |M(h, \gamma^* \to X)|^2 = \frac{2}{\pi} \, \mathrm{Im}\{M(h, \gamma^* \to h, \gamma^*)\}. \tag{2.2}$$

Then, thanks to the reduction formula, we can write the amplitude $M(h, \gamma^* \to h, \gamma^*)$ as the Fourier transform of the following matrix element:

$$\begin{aligned} M(h(P), \gamma^*(q) \to h(P), \gamma^*(q)) &= \langle h(P), \gamma^*(q) | h(P), \gamma^*(q) \rangle = \\ &= \epsilon_\mu(q)\epsilon_\nu(q) i \int d^4x \, e^{iqx} \, \langle h(P) | J^\mu(x) J^\nu(0) | h(P) \rangle = \epsilon_\mu(q)\epsilon_\nu(q) W^{\mu\nu}, \end{aligned} \tag{2.3}$$

where

$$W^{\mu\nu} := i \int d^4x \, e^{iqx} \, \langle h(P) | J^\mu(x) J^\nu(0) | h(P) \rangle \, . \tag{2.4}$$

The $Q^2 \to \infty$ limit in momentum space, corresponds $x^2 \to 0$ limit in conjugate space. Therefore, we can apply the OPE (Operator Product Expansion or Wilson

expansion) to the operator pair, which can be written as

$$J^\mu(x)J^\nu(0) = \sum_{N=0}^{\infty} C_N(x^2) O_N^{\mu\nu\alpha_1\ldots\alpha_n} x_{\alpha_1\ldots\alpha_n}, \tag{2.5}$$

where $C_N(x^2)$ are position dependent coefficients and $\{O_N\}$ are a basis of the space of operators. It is important to highlight the fact that we are considering the light cone limit $x^2 \to 0$, which does not imply that every component $x^\mu \to \infty$. Similarly, as $Q^2 \to \infty$, the components of the 4-momentum q^μ are not approaching ∞. Therefore, in a DIS process we are interested in the high energy limit, but we are free to ask for the ratio $x := \frac{Q^2}{2(P,q)} \propto q^\mu$ to be finite.

As regards the operator basis $\{O_N\}$, since the dimension of the product $J^\mu(x)J^\nu(0)$ in mass units is a constant, the dimension of the rhs of equation (2.5) must be constant, too. Therefore, if O_N has dimension d_O, then its coefficient C_N must have dimension $d - d_O$. Since $C_N(x^2)$ depends only on one dimension variable x^2, we get that $C_N(x^2) \propto \left(\frac{1}{x^2}\right)^{d-d_O}$, hence the operator O_N is multiplied by a suppression factor of $\left(\frac{1}{Q}\right)^{d_O}$. Moreover, if $O_N^{\mu\nu\alpha_1\ldots\alpha_n}$ is a s-spin operator, its matrix element $\langle h(P)|O^{\mu\nu\alpha_1\ldots\alpha_n}|h(P)\rangle$ will carry s additional factors $\left(\frac{2(P,q)}{Q^2}Q\right)^s$, hence an enhancing factor Q^s. Therefore, globally, the factor is $\left(\frac{1}{Q}\right)^{d_O-s}$, which means that the leading contributions of the OPE come from the operators with lowest twist $t = d_O - s$.

It can be shown that the lowest allowed value for QCD is $t = 2$, and the leading contributions come from the following elements of the operator basis:

$$O_N^{\alpha_1\ldots\alpha_N} := \bar{\psi}\gamma^{(\alpha_1}D^{\alpha_2}\ldots D^{\alpha_N)}\psi - \text{trace terms}, \tag{2.6}$$

where the symmetrization $(\alpha_1\ldots\alpha_N)$ and the subtraction of the traces are aimed to extract the highest spin components. If we substitute these basis elements into the Fourier transform of the operator product, we get the following operator approximation:

$$i\int d^4x\, e^{iqx} J^\mu(x)J^\nu(0) \approx 4\sum_{N=2}^{\infty} \frac{2q_{\alpha_1}\ldots 2q_{\alpha_{N-2}}}{(Q^2)^{N-1}} O_N^{\mu\nu\alpha_1\ldots\alpha_{N-2}}. \tag{2.7}$$

In order to obtain $W^{\mu\nu}$, our task is now to compute the mean value of this operator basis elements on the state $|h(P)\rangle$. Since the operator has N 4-vector indices, and the matrix element can depend only on the 4-vector P, then the matrix element can be written as follows:

$$\langle h(P)|O_N^{\mu\nu\alpha_1\ldots\alpha_{N-2}}|h(P)\rangle = 2P^\mu P^\nu P^{\alpha_1}\ldots P^{\alpha_{N-2}} f_N. \tag{2.8}$$

If we substitute the form (2.8) into definition (2.4), we get:

$$W^{\mu\nu} = 8\sum_N \left(\frac{2(q,P)}{Q^2}\right)^{N-2} \frac{P^\mu P^\nu}{Q^2} f_N C_N(Q^2), \tag{2.9}$$

where we can recognise the definition of the Bjorken variable (2.30), obtaining

$$W^{\mu\nu}(x,Q^2) = 8\sum_N \left(\frac{1}{x}\right)^{N-2} \frac{P^\mu P^\nu}{Q^2} f_N C_N(Q^2). \tag{2.10}$$

In expression (2.10), we can focus on the scalar part

$$W(x,Q^2) := \sum_N \left(\frac{1}{x}\right)^{N-2} f_N C_N(Q^2), \tag{2.11}$$

where f_N is the only term which depends on the hadron under consideration. As it has already been stated, the OPE is a relation between operators, therefore it does not depend on the state where the operators are to be evaluated. Therefore, we now exploit this *universality* characteristic, and we choose a convenient state where the matrix element f_N takes the value of 1 at tree level, in order to compute the coefficients $C_N(Q^2)$. This state is obviously the single quark state $|q(P)\rangle$. We now define

$$\hat{W}^{\mu\nu} := i\int d^4x\, e^{iqx} \langle q(P)|J^\mu(x)J^\nu(0)|q(P)\rangle\,, \tag{2.12}$$

and after rerunning the same argument we get:

$$\hat{W}^{\mu\nu}(x,Q^2) = 8\sum_N \left(\frac{1}{x}\right)^{N-2} \frac{P^\mu P^\nu}{Q^2} C_N(Q^2) \tag{2.13}$$

and its scalar component

$$\hat{W}(x,Q^2) := \sum_N \left(\frac{1}{x}\right)^{N-2} C_N(Q^2). \tag{2.14}$$

Now, our aim is to compute the coefficients $C_N(Q^2)$ from the relation (2.14). This can be done by realising that (2.14) is a Laurent expansion, therefore by defining $\omega = 1/x$, the coefficients of this series can be obtained thanks to the residue theorem as follows:

$$C_N(Q^2) = \int_{|\omega|=r} \frac{d\omega}{2\pi i} \frac{\hat{W}(\omega,Q^2)}{\omega^N}, \tag{2.15}$$

where $\hat{W}(\omega,Q^2)$ is intended as its analytic continuation for complex values of ω, and r has to be chosen sufficiently small in order to not encounter points where $\hat{W}$ is not analytic. We now ask ourselves where the non-analiticity points are. The answer lies in the consideration of section 2.4 that the physical region of DIS is represented by $x \in [0,1]$, i.e. $\omega > 1$. Since $\hat{W}^{\mu\nu}$ is a 2-point correlation function, then it admits a Kallen-Lehmann spectral representation, which implies that it present a branch cut in the physical region $\omega > 1$ along the real axe. Moreover, by recalling the definition (2.12), it is clear that $\hat{W}^{\mu\nu}(x,Q^2)$ must be symmetric under $x \mapsto -x$, therefore $\omega < -1$ is a branch cut along the real axe, too, and no more non-analytic regions can be present. Then, we can deform the integration path into the one of figure 2.1 without changing the value of the integral. The only contributions to this integral come from the paths along the branch cut and,

exploiting the Schwarz's reflection principle and the symmetric property of $\hat{W}$, we can obtain the coefficients as:

$$C_N(Q^2) = 4i \int_0^1 \frac{dx}{2\pi i} x^{N-2} \hat{W}(x, Q^2) = \int_0^1 dx\, x^{N-1} \frac{2}{\pi} \frac{\mathrm{Im}\Big\{\hat{W}(x, Q^2)\Big\}}{x}, \tag{2.16}$$

where $C_N(Q^2)$ are the Mellin transform of the function

$$C(x, Q^2) := \frac{2}{\pi} \frac{\mathrm{Im}\Big\{\hat{W}(x, Q^2)\Big\}}{x}. \tag{2.17}$$

This relation is clearly nothing but the optical theorem for the partonic cross section. By retaining only the most singular term in the expansion and substituting our definitions into the eq. (2.2), we get:

$$\sigma_{tot}(N, Q^2) = f_N C_N(Q^2). \tag{2.18}$$

By denoting the inverse-Mellin transform of f_N with

$$f(x) = \int_{c-i\infty}^{c+i\infty} dN\, x^{-N} f_N, \tag{2.19}$$

and by computing the inverse-Mellin transform of equation (2.18), the result is:

$$\sigma(x, Q^2) = (f \otimes C)(x, Q^2). \tag{2.20}$$

The symbol $\otimes$ denotes the convolution integral. Its definition and its properties under Mellin transformations are reported in appendix A.1. If we restore the different parton indices (quark flavours and gluon) in the above formula we obtain:

$$\sigma(N, Q^2) = \sum_i f_i(N) C_i(N, Q^2), \tag{2.21}$$

where we can read that, in general, different i partons (quarks and gluon) are associated with a different function f_i.

The matrix $f_i(x)$ has the physical interpretation of the mean presence of a parton in a hadron state, and the Bjorken variable x has the interpretation of the fraction of the 4-momentum of the hadron carried by the quark. $f_f(x)$ is called *parton distribution function* (PDF), they have no dependence on the considered process and can be measured by fitting experimental data.

2.2 Higher order factorization

The derivation of factorization of section 2.1 is correct only at tree level and in this section we state how it can be generalised to include higher order corrections. It is important to highlight that we exploited the fact that the matrix element f_N takes the value 1 at tree level when evaluated in a single quark state. If we consider higher order corrections, we must also compute loop diagrams, which are divergent. These singularities are nothing but the initial state collinear singularities we discussed

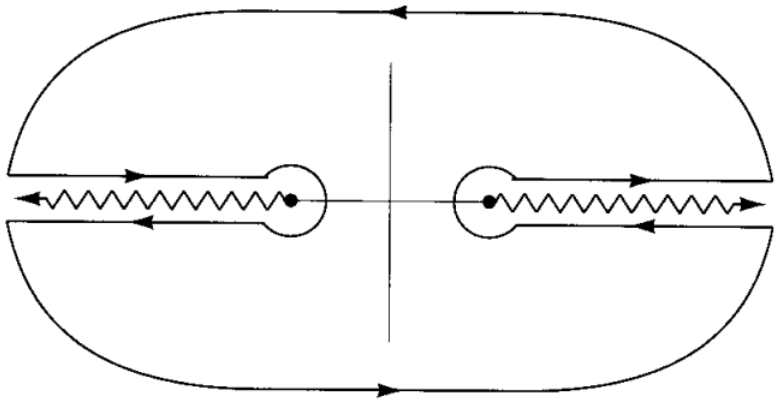


Figure 2.1: The figure illustrates the analytic properties of the complex function $\hat{W}(\omega, Q^2)$ and its integration path.

in section 1.8.2. Since PDFs are experimentally measured quantities, to remove these divergences, we should follow the renormalization procedure, for example in dimensional regularization. By defining a multiplicative counter term $Z(\mu_F^2, \epsilon)$, we can define a renormalized PDF $f^{ren}(N, \mu_F^2) := f(N, \epsilon)Z(\mu_F^2, \epsilon)$. Therefore, both the PDF $f(x)$ and the partonic cross section $C(x, Q^2)$ acquire a dependence on a new scale μ_F^2, which is called *factorization scale*. Then, the factorization in Mellin space reads as follows:

$$\sigma(N, Q^2) = \sum_i f_i(N, \mu_F^2) C_i(N, Q^2, \mu_F^2), \tag{2.22}$$

where, since μ_F^2 is an arbitrary scale, the physical observable $\sigma(N, Q^2)$ can not depend on it.

Moreover, the PDFs dependence on the factorization scale can be computed. In fact, by imposing the Callan-Symanzik equation of section 1.5 for a generic observable, it can be shown that the solutions are the DGLAP equations (see [2]), which read as follows:

$$\frac{df_i(x, \mu_F^2)}{d \ln \mu_F^2} = \sum_j (P_{j\to i} \otimes f_j)(x, \mu_F^2), \tag{2.23}$$

or more explicitly

$$\frac{df_i(x, \mu_F^2)}{d \ln \mu_F^2} = \sum_j \int_x^1 \frac{d\xi}{\xi} P_{j\to i}\left(\frac{x}{\xi}, \alpha_s(\mu_F^2)\right) f_j(\xi, \mu_F^2), \tag{2.24}$$

where $P_{j\to i}$ are the splitting functions, and we recall the fact that i, j indices include quark flavours and gluon.

2.3 PDFs properties

In this section we state some relations that the PDFs must satisfy, in order to be a proper description of the considered hadron.

Firstly, since the Bjorken variable x is the fraction of momentum carried by the parton involved in the process, the sum over all partons integrated over this variable must result in the total momentum of the hadron. This constraint reads as follows:

$$1 = \sum_i \int_0^1 dz\, z f_i(z). \tag{2.25}$$

Moreover, PDFs from different hadrons can be distinguished because they must verify different relations. In fact, for instance for the proton the quark PDFs must satisfy:

$$\int_0^1 dz\, f_u(z) - f_{\bar{u}}(z) = 2, \tag{2.26}$$

$$\int_0^1 dz\, f_d(z) - f_{\bar{d}}(z) = 1, \tag{2.27}$$

while, for other flavours c

$$\int_0^1 dz\, f_c(z) - f_{\bar{c}}(z) = 0. \tag{2.28}$$

These relations can be interpreted as the mean presence of a quark of a certain flavour inside the proton. The quark flavours with non-zero PDFs integrated value (in the sense of the relations above) are called *valence* quarks.

If we consider a process with multiple incoming hadrons, then the PDFs of each hadron must individually verify the above relations.

2.4 DIS and hadron collision

In this section we present the DIS and hadron collision cross sections expressed with the most commonly used kinematic variables.

If we consider a DIS process, i.e.

$$h(P) + l(q) \longrightarrow l(q') + X(P_X), \tag{2.29}$$

(l lepton) the cross section is usually expressed in terms of the Bjorken variable

$$x = \frac{Q^2}{2(p,q)}, \tag{2.30}$$

where $Q^2 = -q^2$ and $x \in [0,1]$ in the physical kinematic region. If we denote with z the fraction of 4-momentum of the hadron carried by the parton, then (2.22) can be rewritten as

$$\sigma(x, Q^2) = \sum_i \int_x^1 dz\, f_i(z, \mu_F^2) \hat{\sigma}_i \left(\hat{x} = \frac{x}{z}, Q^2, \mu_F^2 \right). \tag{2.31}$$

Analogously, also for the case of hadron collision with a massive final state (for example a Higgs boson), i.e.

$$h_1(P_1) + h_2(P_2) \longrightarrow H(P_H) + X(P_X), \tag{2.32}$$

the usual variables are the mass of the final state particle $M_H^2 = P_H^2$, the fraction of the 4-momenta of the hadrons carried by each parton $p_i = x_i P_i$, the hadronic centre of mass energy $s = (P_1 + P_2)^2$, the partonic centre of mass energy $\hat{s} = (p_1 + p_2)^2$, the fraction of the hadronic energy used to produce the final state particle $\tau = \frac{M_H^2}{s}$ and the fraction of the partonic energy used to produce the final state particle $z = \frac{M_H^2}{\hat{s}} = \frac{\tau}{x_1 x_2}$.

In section 2.1, we proved factorization for a DIS process, i.e. with only one incoming hadron. The generalisation to the case of two incoming incoming hadrons, for instance the case of a hadron collision, is very technical and it goes beyond the aim of this work. Therefore, we will simply assume that for a hadron collision the factorization hypothesis holds. The cross section in Mellin space takes the following form:

$$\sigma(N, Q^2) = \sum_{ij} f_i(N, \mu_F^2) f_j(N, \mu_F^2) C_{ij}(N, Q^2, \mu_F^2). \tag{2.33}$$

As for DIS, by computing the inverse-Mellin transform, we obtain:

$$\sigma(\tau, M_H^2) = \sum_{ij} (f_i \otimes f_j \otimes C_{ij})(\tau, M_H^2). \tag{2.34}$$

In terms of the variables above, it can be rewritten as

$$\sigma(\tau, M_H^2) = \sum_{ij} \int_\tau^1 dx_1\, f_i(x_1, \mu_F^2) \int_{\tau/x_1} dx_2\, f_j(x_2, \mu_F^2) C_{ij}\left(z = \frac{\tau}{x_1 x_2}, M_H^2, \mu_F^2\right). \tag{2.35}$$

The convolution can be also factored by defining the *parton density luminosity* $\mathcal{L}_{ij}(x) \coloneqq (f_i \otimes f_j)(x)$, therefore we can obtain the hadronic cross section as the convolution of the parton density luminosity with the partonic cross section, i.e.

$$\sigma(\tau, M_H^2) = \sum_{ij} (\mathcal{L}_{ij} \otimes C_{ij})(\tau, M_H^2). \tag{2.36}$$

Chapter 3

Resummation

In section 1.8 we discussed the emergence of IR singularities and their cancellation mechanism. In this chapter we focus on the leftovers of these divergences. In fact, these residues, in the limit of soft emissions, become big logarithms at all orders, spoiling the perturbative series. We now provide a description of the origin and of the form of these enhanced logarithms and then we prove that they are mapped into big logs also in Mellin space. Afterwards, we briefly present an argument to show that the resummed expression takes the form of an exponential series in the single and multi-scale case. Finally, we describe how the coefficients of this series can be determined by matching the resummed expression with the fixed order cross section.

3.1 Logarithms

In this section we describe the origin of enhanced logarithms. Firstly, we analyse the collinear logarithms, then we consider logarithms due to soft emission and our they are mapped into logs in Mellin space.

3.1.1 Collinear logarithms

To explain the origin of collinear logarithms, we consider the NLO cross section of DIS, already discussed in section 1.8.2. Here we report the partonic cross section (1.40):

$$C^{NLO} = \frac{\alpha_s(Q^2)}{2\pi} \int_{\mu^2}^{(k_T^2)^{max}} \frac{dk_T^2}{k_T^2} \int_\tau^1 d\tau \, P_{q\to q}(\tau)\sigma^{LO}(zp). \tag{3.1}$$

In a DIS, it is easy to check that the maximum transverse momentum available for the emitted gluon is $(k_T^2)^{max} = \frac{(s-Q^2)^2}{s}$, which can be written as $(k_T^2)^{max} = Q^2\frac{(1-\tau)^2}{\tau}$ in terms of $\tau = \frac{Q^2}{s}$. Therefore, if we compute the transverse momentum integral, we obtain the following logarithmic factor:

$$\alpha_s(Q^2)\ln\left(\frac{Q^2}{\mu^2}\frac{(1-\tau)^2}{\tau}\right) = \alpha_s(Q^2)\ln\left(\frac{Q^2}{\mu^2}\right) + \alpha_s(Q^2)\ln\left(\frac{(1-\tau)^2}{\tau}\right). \tag{3.2}$$

It is important to remark that, in the example above, we considered a DIS process, but for the case of hadron collision we would have obtained the same logarithmic

factor, with the only difference that in the $(k_T^2)^{max}$ integration endpoint we should have substituted the Q^2 with the mass M^2 of the final state particle. Q^2 or M_H^2 are usually called the *hard scale* of the process.

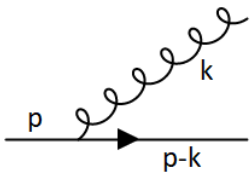
The first term $\alpha_s(Q^2)\ln(Q^2/\mu^2)$ is $O(1)$ at high energy because the logarithm compensates the α_s decrease. In general, if we consider multiple collinear emission, we obtain terms of the form $\alpha_s^n \ln^n(Q^2/\mu^2)$, which are again $O(1)$, hence at all orders we obtain terms of the same order and the perturbative serie is spoiled. Therefore, it is convenient to introduce the notation LL ($\alpha^k \ln^k$ leading log), NLL ($\alpha^{k+1}\ln^k$ next to leading log), ... and to reorganise the perturbative series by collecting all the terms in the series which are of the same order. For the case of $\ln(Q^2/\mu^2)$, the task of resummation is performed by DGLAP equations (2.23), which enable us to subtract collinear logarithms.

The second logarithm of eq. (3.2) grows as τ is approaching 1, i.e. in the soft limit. Moreover, by looking at the inner integral over the variable z, the splitting function contains a plus distribution $\frac{1}{(1-z)_+}$. Therefore, by computing the integral, we obtain another logarithmic contribution of the form $\ln(1-\tau)$, which is enhanced in the soft limit. Therefore, overall the emission provided us with a double logarithm of soft-collinear origin. Depending on the factorization scheme, we can choose to subtract from the partonic cross section not only $\ln(Q^2/\mu^2)$, but also $\ln(1-\tau)^2$, and include it in the PDFs. In this work we opt for the *minimal subtraction scheme* ($\overline{MS}$), which means that we only subtract $\ln(Q^2/\mu^2)$ from the partonic cross section.

3.1.2 Soft logarithms

In this section we aim to explain the origin of purely soft logarithms. In section 1.8.1, we described how the cross section, which involves the emission of a real soft gluon, factors in the product of an Eikonal factor and the cross section without the emission. The Eikonal factor is

$$E(p;k,p-k) = \quad = \frac{p^\mu}{(p,k)}, \tag{3.3}$$



where k is the 4-momentum of the emitted gluon and p the 4-momentum of the quark which emitted it. In the soft limit, the emitted gluon is also becoming collinear, hence by defining $k=(1-z)p$, the soft limit is obtained for $z\to 1$ and the Eikonal factor is proportional to $\frac{1}{1-z}$. Therefore the contribution to the cross section takes the following form:

$$\sigma_{real}^{NLO} \propto \int_\tau^1 \frac{dz}{1-z}\sigma^{LO}(zp). \tag{3.4}$$

On the other hand, if we consider the corresponding soft virtual gluon contribution to the cross section, we obtain:

$$\sigma_{virt}^{NLO} \propto \int_\tau^1 \frac{dz}{1-z}\sigma^{LO}(p), \tag{3.5}$$

and by adding virtual and real contributions together we get

$$\sigma^{NLO} \propto \int_\tau^1 \frac{dz}{(1-z)_+} \sigma^{LO}(zp). \tag{3.6}$$

By computing the above integral, we obtain a factor $\ln(1-\tau)$, which is again big as $\tau \to 1$.

Moreover, if we consider the emission of n gluons, we obtain n factors $\frac{1}{(1-z)_+}$. By integrating $n-1$ of them and leaving the n-th one, the leading contribution takes the following form:

$$\left[\frac{\ln^{n-1}(1-z)}{1-z}\right]_+ . \tag{3.7}$$

3.1.3 Logarithms in Mellin space

Since the cross section factorizes in Mellin space, we are interested in how contributions of the form (3.7) are mapped in Mellin space. In this section we show that distributions (3.7) in the soft region in conventional space are mapped into powers of $\ln(N)$ for $N \in \mathbb{C}$ and $|N| \to \infty$ in Mellin space. In fact, by considering the Mellin transform of a generic function

$$\int_0^1 dz\, z^{N-1} f(z), \tag{3.8}$$

the soft region $z \to 1$ corresponds to the limit $|N| \to \infty$, because for $z < 1$, the function is mapped to 0. The same argument can be applied to the inverse Mellin transform, hence this is a $1 \leftrightarrow 1$ correspondence.

Now, if we consider the Mellin transform of the contributions which need to be resummed, i.e.

$$I_p := \int_0^1 dz\, z^{N-1} \left(\frac{\ln^p(1-z)}{1-z}\right)_+ = \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \ln^p(1-z), \tag{3.9}$$

by defining the generating functional

$$G(N,\eta) := \int_0^1 dz\, (z^{N-1}-1)(1-z)^{\eta-1}, \tag{3.10}$$

we can obtain our contributions as:

$$I_p = \left.\frac{d^p}{d\eta^p}\right|_{\eta=0} G(N,\eta). \tag{3.11}$$

In the definition of the generating functional we can recognise the Euler beta function definition, i.e.

$$G(N,\eta) = B(N,\eta) - \frac{1}{\eta} = \frac{\Gamma(N)\Gamma(\eta)}{\Gamma(N+\eta)} - \frac{1}{\eta}. \tag{3.12}$$

By taking the first derivative of expression (3.12) as in relation (3.11), it is easy to compute:

$$I_1 = \frac{1}{2}\left[\left(\psi^{(0)}(N) + \gamma_E\right)^2 + \zeta_2 - \psi^{(1)}(N)\right], \tag{3.13}$$

where γ_E is Euler-Mascheroni constant, ζ_2 denotes the Riemann zeta function evaluated at 2, and $\psi^{(n)}(x)$ is the n-th polygamma function, which is defined as follows:

$$\psi^{(n)}(x) := \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x). \tag{3.14}$$

In the large-N limit, it can be shown that $\psi^{(0)}(N) \sim \ln N + O(1/N)$, hence $I_1 \sim \frac{1}{2}\ln^2(N) + \gamma_E \ln N + O(1)$. In general, the asymptotic behaviour of I_p is the following:

$$I_p \sim \frac{1}{p+1} \ln^{p+1}\left(\frac{1}{N}\right) - \gamma_E \ln^p\left(\frac{1}{N}\right) + O\left(\ln^{p-1}\left(\frac{1}{N}\right)\right). \tag{3.15}$$

Thanks to the fact that logarithms are mapped into logarithms via Mellin transform, the notation LL, NLL, ... can be immediately adopted also in Mellin space.

3.2 Soft resummation

In this section we derive the resummation formula for the cross section of a Drell-Yan or a Higgs production processes, i.e. processes with a colourless massive final state. The mass of the final state particle is denoted with M_H^2 and it is the hard scale of the process, as discussed in section 3.1.1. By changing the hard scale $M_H^2 \mapsto Q^2$ and the soft scale $M_H^2(1-x)^2 \mapsto Q^2(1-x)$, the following argument can be also applied to the case of DIS (for the interested reader we refer to [13]). The following derivation is based on [13] and [15]. We start by considering the factorization in Mellin space for hadron collisions (we drop parton indices, i.e. flavours and gluon, for simplicity):

$$\sigma(N, M_H^2) = \mathcal{L}(N, \mu_F^2) C\left(N, \frac{M_H^2}{\mu_R^2}, \frac{\mu_R^2}{\mu_F^2}, \alpha_s(\mu_R^2)\right), \tag{3.16}$$

where $\mathcal{L}(N, \mu_F^2)$ is the parton density luminosity defined in section 2.4. Since both the renormalization scale μ_R and the factorization scale μ_F are arbitrary, then we can choose them to be equal to each other $\mu_R = \mu_F = \mu$. Hence the equation (3.16) becomes:

$$\sigma(N, M_H^2) = \mathcal{L}(N, \mu^2) C\left(N, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right), \tag{3.17}$$

where the cross section σ is μ-independent because μ^2 is an arbitrary scale and σ is an observable. It is now convenient to define the *physical anomalous dimension* of the cross section:

$$\gamma(N, \alpha_s(M_H^2)) := \frac{d}{d\ln M_H^2} \ln \sigma(N, M_H^2). \tag{3.18}$$

Obviously, since $\sigma(N, M_H^2)$ is μ-independent, $\gamma(N, \alpha_s(M_H^2))$ can not depend on μ either. Now, by substituting $\sigma(N, M_H^2)$ from eq. (3.17) into the definition (3.18) and by exploiting the fact that $\mathcal{L}(N, \mu^2)$ is M_H-independent, we obtain:

$$\gamma(N, \alpha_s(M_H^2)) = \frac{d}{d\ln M_H^2} \ln C\left(N, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right), \tag{3.19}$$

therefore $\gamma(N, \alpha_s(M_H^2))$ is also the physical anomalous dimension of the coefficient function $C(N, M_H^2/\mu^2, \alpha_s(\mu^2))$. Finally, by solving eq. (3.19), we can write the resummed coefficient function:

$$C\left(N, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C(N, 1, \alpha_s(\mu^2)) \exp\left\{\int_{\mu^2}^{M_H^2} \frac{dk^2}{k^2} \gamma(N, \alpha_s(k^2))\right\}. \tag{3.20}$$

We now recall the fact that the renormalized coefficient function C is multiplicative renormalized as follows:

$$C\left(N, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = Z^C(N, \alpha_s(\mu^2), \epsilon) C^0(N, M_H^2, \alpha_s^0, \epsilon), \tag{3.21}$$

where C^0 is the bare coefficient function, α_s^0 is the bare coupling constant and we choose to regularize the divergent expression in dimensional regularization with $d = 4 - 2\epsilon$. Moreover, thanks to dimensional analysis, we can deduce that the bare coefficient function in d dimensions can depend on M_H^2 and α_s^0 only through the combination $M_H^{-2\epsilon}\alpha_s^0$, i.e.

$$C^0(N, M_H^2, \alpha_s^0, \epsilon) = C^0(N, M_H^{-2\epsilon}\alpha_s^0, \epsilon). \tag{3.22}$$

Therefore, if we substitute the relation (3.21) into eq. (3.19), and by exploiting the facts that Z^C is M_H-independent and C^0 depends only on $M_H^{-2\epsilon}\alpha_s^0$, we obtain:

$$\gamma(N, \alpha_s(M_H^2)) = -\epsilon\alpha_s^0 \frac{d}{d\ln\alpha_s^0} \ln C^0(N, M_H^2, \alpha_s^0, \epsilon). \tag{3.23}$$

We now make the hypothesis that the bare coefficient function C^0 in Mellin space factorizes in the product of two bare coefficient functions $C^{(c)0}(M_H^2, \alpha_s^0, \epsilon)$ and $C^{(l)0}(N, M_H^2, \alpha_s^0, \epsilon)$, where the former represents the virtual gluons emission contributions and the latter the real gluons emission contributions. $C^{(c)0}$ does not depend on N because the virtual contributions have Born kinematic. On the other hand, $C^{(l)0}$ must depend on N because it contains real emission contribution, and in particular, thanks to a phase space structure argument, [13] shows that $C^{(l)0}$ can depend on M_H and z only through the combination $M_H^2(1-z)^2$. It also shows that in Mellin space the $M_H^2(1-z)^2$ dependence is converted into a M_H^2/N^2 dependence. Therefore the factorization reads as follows:

$$C^0(N, M_H^2, \alpha_0, \epsilon) = C^{(c)0}(M_H^2, \alpha_0, \epsilon) C^{(l)0}\left(\frac{M_H^2}{N^2}, \alpha_0, \epsilon\right). \tag{3.24}$$

By substituting expression (3.24) in eq. (3.23), we obtain the following equation:

$$\gamma(N, \alpha_s(M_H^2)) = \gamma^c\left(\frac{M_H^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) + \gamma^l\left(\frac{M_H^2/N^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right), \tag{3.25}$$

where we defined

$$\gamma^c\left(\frac{M_H^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) := -\epsilon\alpha_s^0 \frac{d}{d\ln\alpha_s^0} \ln C^{(c)0}(M_H^2, \alpha_s^0, \epsilon) \tag{3.26}$$

and

$$\gamma^l\left(\frac{M_H^2/N^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) := -\epsilon\alpha_s^0\frac{d}{d\ln\alpha_s^0}\ln C^{(l)0}\left(\frac{M_H^2}{N^2},\alpha_s^0,\epsilon\right). \tag{3.27}$$

It is important to remark that γ^c and γ^l are not individually finite for $\epsilon\to 0$, but their sum, i.e. γ, is by definition finite. Moreover, since γ is renormalization-group invariant, i.e. μ-independent, by deriving both members of eq. (3.25) with respect to μ we obtain:

$$\begin{cases}\lim_{\epsilon\to 0}\frac{d}{d\ln\mu^2}\gamma^l\left(\frac{M_H^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = -\hat{g}(\alpha_s(\mu^2))\\ \lim_{\epsilon\to 0}\frac{d}{d\ln\mu^2}\gamma^c\left(\frac{M_H^2/N^2}{mu^2},\alpha_s(\mu^2),\epsilon\right) = \hat{g}(\alpha_s(\mu^2)),\end{cases} \tag{3.28}$$

where $\hat{g}(\alpha_s(\mu^2))$ is a finite power series in the coupling constant $\alpha_s(\mu^2)$.

Therefore, by solving the renormalization-group equations for γ^c and γ^l and by adding the two solutions, we can now write the physical anomalous dimension γ in terms of the series $\hat{g}$ as follows:

$$\gamma(N,\alpha_s(M_H^2)) = \hat{g}^c(\alpha_s(M_H^2)) + \int_{M_H^2}^{M_H^2/N^2}\frac{d\lambda^2}{\lambda^2}\hat{g}(\alpha_s(\lambda^2)), \tag{3.29}$$

where $\hat{g}^c$ is a power series in the coupling constant $\alpha_s(M_H^2)$. If we substitute this expression for γ into eq. (3.20), we obtain the following resummation formula:

$$C\left(N,\frac{M_H^2}{\mu^2},\alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2},\alpha_s(M_H^2)\right)\exp\left\{\int_{\mu^2}^{M_H^2}\frac{dk^2}{k^2}\int_{k^2}^{k^2/N^2}\frac{d\lambda^2}{\lambda^2}\hat{g}(\alpha_s(\lambda^2))\right\}, \tag{3.30}$$

where C^c is a power series in the coupling constant $\alpha_s(M_H^2)$. Here we report two equivalent forms of the resummation formula (3.30):

$$C\left(N,\frac{M_H^2}{\mu^2},\alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2},\alpha_s(M_H^2)\right)\exp\left\{\int_1^{N^2}\frac{dn}{n}\int_{n\mu^2}^{M_H^2}\frac{dk^2}{k^3}\hat{g}(\alpha_s(k^2/n))\right\} \tag{3.31}$$

and

$$C\left(N,\frac{M_H^2}{\mu^2},\alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2},\alpha_s(M_H^2)\right)$$
$$\exp\left\{2\int_0^1\frac{z^{N-1}-1}{1-z}\int_{\mu^2}^{M_H^2(1-z)^2}\frac{dk^2}{k^2}g(\alpha_s(k^2))\right\}, \tag{3.32}$$

where $g(\alpha_s(k^2))$ is also a power series in the coupling constant.

3.3 The multi-scale case

In this section we generalise the argument of section 3.2 to the case of a process with two generic hard scales and two generic soft scales in order to obtain the

resummation formulas. The derivation is based on [15]. We consider a process with two hard scales denoted with Q_1^2 and Q_2^2, two soft/collinear scales $\Lambda_1^2(Q_1^2, N)$ and $\Lambda_2^2(Q_2^2, N)$, and we assume the factorization of the bare coefficient function in a product of virtual emission contributions and emission contributions with respect to the two hard scales:

$$C^0(N, Q_1^2, Q_2^2, \alpha_s^0, \epsilon) = \\ C^{(c)0}(N, Q_1^2, Q_2^2, \alpha_s^0, \epsilon) C^{(l_1)0}(\Lambda_1^2(Q_1^2, N), \alpha_s^0, \epsilon) C^{(l_2)0}(\Lambda_2^2(Q_2^2, N), \alpha_s^0, \epsilon). \quad (3.33)$$

By rerunning the same argument of the previous section, we find the following resummation formula:

$$C\left(N, \frac{Q_1^2}{\mu^2}, \frac{Q_2^2}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{Q_1^2}{\mu^2}, \frac{Q_2^2}{\mu^2}, \alpha_s(\mu^2)\right) \\ \exp\left\{\int_1^{N^a} \frac{dn}{n} \int_{n\mu^2}^{Q_1^2} \frac{dk^2}{k^2} \hat{g}_1(\alpha_s(k^2/n)) + \int_1^{N^b} \frac{dn}{n} \int_{n\mu^2}^{Q_2^2} \frac{dk^2}{k^2} \hat{g}_2(\alpha_s(k^2/n))\right\}, \quad (3.34)$$

or equivalently

$$C\left(N, \frac{Q_1^2}{\mu^2}, \frac{Q_2^2}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{Q_1^2}{\mu^2}, \frac{Q_2^2}{\mu^2}, \alpha_s(\mu^2)\right) \\ \exp\left\{\int_0^1 \frac{z^{N-1}-1}{1-z} \left(\int_{\mu^2}^{\Lambda_1(Q_1^2,z)} \frac{d\lambda^2}{\lambda^2} g_1(\alpha_s(\lambda^2)) + \int_{\mu^2}^{\Lambda_2(Q_2^2,z)} \frac{d\lambda^2}{\lambda^2} g_2(\alpha_s(\lambda^2))\right)\right\}. \quad (3.35)$$

In [15] it is shown the application of this formalism to the case of transverse momentum distributions. In that case, the two hard scales are $Q_1^2 = M_H^2$ and $Q_2^2 = M_H p_t$ and there are soft emissions with respect to the first scale, hence the soft scale is $\Lambda_1^2(Q_1^2) = M_H^2/N^2$, and collinear emissions with respect to the second scale, hence the collinear scale is $\Lambda_2^2(Q_2^2) = M_H p_T/N$. Therefore the resummation formula takes the following form:

$$C\left(N, \frac{M_H^2}{\mu^2}, \frac{M_H p_T}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2}, \frac{M_H p_T}{\mu^2}, \alpha_s(\mu^2)\right) \\ \exp\left\{\int_1^{N^2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} \hat{g}_1(\alpha_s(k^2/n)) + \int_1^{N} \frac{dn}{n} \int_{n\mu^2}^{M_H p_T} \frac{dk^2}{k^2} \hat{g}_2(\alpha_s(k^2/n))\right\}, \quad (3.36)$$

or equivalently

$$C\left(N, \frac{M_H^2}{\mu^2}, \frac{M_H p_T}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2}, \frac{M_H p_T}{\mu^2}, \alpha_s(\mu^2)\right) \\ \exp\left\{\int_0^1 \frac{z^{N-1}-1}{1-z} \left(\int_{\mu^2}^{M_H^2(1-z)^2} \frac{d\lambda^2}{\lambda^2} g_1(\alpha_s(\lambda^2)) + \int_{\mu^2}^{M_H p_T(1-z)} \frac{d\lambda^2}{\lambda^2} g_2(\alpha_s(\lambda^2))\right)\right\}. \quad (3.37)$$

In section 4.6 and 4.7 the same formalism will be applied to produce the resummation formula for rapidity distributions in the singly and doubly soft limit.

3.4 Matching procedure

In the previous sections, we proved that soft perturbative corrections must exponentiate. What is still to be determined are the coefficients of the power series in the exponent (g_1 and g_2 in our notation). Therefore, we can take our resummation formula and compute them to a fixed order in perturbation theory: this computation must bring us back to the fixed order computation (in Mellin space). Obviously, the obtained expression must be equal to the fixed order computation of the partonic cross section (in Mellin space), therefore we can get the coefficients of our power series by comparing the 2 expression, and then substitute them back into our all order resummation formulas. Now that we know how to obtain the resummation formulas, but these formulas are valid only in the threshold region, while far away from these kinematic region, we should rely on the fixed order computation. In order to add them together, we must remember to subtract the fixed order expansion of the resummation formula up to the maximum order of the fixed order computation, otherwise these terms would appear twice, which is non-physical.

Chapter 4

Rapidity distribution

In this chapter we apply the formalism introduced in chapter 3 to the case of rapidity distributions. Firstly, we analyse the kinematic structure of rapidity distributions and their factorization in Mellin-Fourier and Mellin-Mellin space. Secondly, we provide the NLO rapidity distributions for Drell-Yan and Higgs production processes and we describe the most commonly used kinematic variables. Thirdly, we describe the kinematic thresholds of such processes, which are the so called *singly* and *doubly* soft limits. These are the regions that must be resummed. Then, we study the phase space structure in order to find out the hard and soft/collinear scales in both the singly and doubly soft cases. Once the scales are known, we will be ready to apply the renormalization-group formalism and derive the resummation formulas.

4.1 Kinematics

In this section we describe the distinction between the hadronic and the partonic processes, the particles involved, the kinematic variables we choose to use and the parametrisation of the particles' 4-momenta. We consider the rapidity distribution $\frac{d\sigma}{dY}$ for the process

$$h_1(P_1) + h_2(P_2) \to H(p_H) + X_H, \tag{4.1}$$

where h_1 and h_2 are the colliding hadrons with four-momenta P_1 and P_2, and H is a massive finale state object (in our case a Higgs boson), whose four-momentum is denoted with p_H. The four-momenta of the hadrons in the hadronic frame of reference are $P_1 = \frac{\sqrt{s}}{2}(1,0,0,1)$ and $P_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)$. The four-momentum p_H is parametrized as

$$p_H = (\sqrt{M_H^2 + |\vec{p}_T|^2}\cosh Y, \vec{p}_T, \sqrt{M_H^2 + |\vec{p}_T|^2}\sinh Y), \tag{4.2}$$

where M_H^2 is the invariant mass of H, $\vec{p}_T$ the transverse momentum, Y the longitudinal rapidity and $s = (P_1 + P_2)^2$ the invariant mass of the colliding hadrons. We also denote with $\tau = \frac{M_H^2}{s}$ the fraction of the hadronic energy required to produce H at rest. The choice of rapidity as a variable instead of the longitudinal momentum itself is due to its adding property under boosts, which will be exploited in what follows.

By assuming that the collision takes place only between two partons, one extracted from each hadron, we now exploit the factorization. Therefore the partonic process is

$$q_i(p_1) + q_j(p_2) \to H(p_H) + X, \tag{4.3}$$

where we denoted with $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$ the four-momenta of the partons involved in the collision, with x_1 and x_2 the fractions of the four-momenta of the hadrons that the partons carry, and with X the extra radiation emitted during this subprocess. In the hadronic frame of reference their four-momenta are $p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1)$ and $p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_2)$, hence the sum of their four-momenta is

$$p_1 + p_2 = \sqrt{\hat{s}}(\cosh\xi, 0, 0, \sinh\xi), \tag{4.4}$$

where we denoted with $\hat{s} = x_1 x_2 s$ the invariant mass of the system composed of the 2 partons and with $\xi = \frac{1}{2}\ln\frac{x_1}{x_2}$ the longitudinal rapidity of this system, which is also the rapidity of the partonic centre of mass in the hadronic frame of reference. In the partonic frame of reference (where the system of the 2 partons is at rest) the four-momenta $p_1 + p_2$, p_H and X can be written as

$$p_1 + p_2 = \sqrt{\hat{s}}(1, 0, 0, 0) \tag{4.5}$$

$$p_H = \left(\sqrt{M_H^2 + p_T^2}\cosh\hat{y}, \vec{p}_T, \sqrt{M_H^2 + p_T^2}\sinh\hat{y}\right) \tag{4.6}$$

$$X = \left(\sqrt{M_X^2 + M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y}}, -\vec{p}_T, -\sqrt{M_H^2 + p_T^2}\sinh\hat{y}\right), \tag{4.7}$$

where $\hat{y} = y - \xi$ is the longitudinal rapidity of H in the partonic frame of reference, thanks to the above mentioned transformation properties of the rapidity under boosts.

4.2 Rapidity distributions factorization

In this section we explain the necessity of a double transform in order to obtain the factorization of the rapidity distribution. Firstly, we describe how to obtain it via a Mellin-Fourier transform, then we explain how we can alternatively obtain factorization by means of a Mellin-Mellin transform. Finally, we make an important remark about how the Mellin-Mellin variables must satisfy a contraint.

4.2.1 Factorization in Mellin-Fourier space

In this section we describe how to obtain factorization in Mellin-Fourier space. The following derivation also sheds light on the choice of the rapidity as a kinematic variable.

Defining $z := \frac{M_H^2}{\hat{s}} = \frac{\tau}{x_1 x_2}$, the rapidity distribution reads as follows:

$$\frac{1}{\tau}\frac{d\sigma}{dY}(\tau, Y, M_H^2) = \sum_{ij}\int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2\, f_i(x_1) f_j(x_2)\frac{dC_{ij}}{d\hat{y}}(z = \frac{\tau}{x_1 x_2}, \hat{y} = Y - \xi, M_H^2), \tag{4.8}$$

where it is straightforward to see that $x_1^0 = \sqrt{\tau}e^Y$ and $x_2^0 = \sqrt{\tau}e^{-Y}$ are the minimum values respectively of x_1 and x_2 in order to produce H at rest in the hadronic frame of reference. With a little abuse of notation, we denote the rapidity distributions only by their dependences, namely

$$\sigma(\tau, Y, M_H^2) := \frac{1}{\tau}\frac{d\sigma}{dy}(\tau, Y, M_H^2) \tag{4.9}$$

$$C_{ij}(z, \hat{y}, M_H^2) := \frac{dC_{ij}}{d\hat{y}}(z, \hat{y}, M_H^2). \tag{4.10}$$

We can rewrite (4.8) by replacing the kinematic constraints $z = \frac{\tau}{x_1 x_2}$ and $\hat{y} = Y - \xi$ with integrals over δ functions, obtaining:

$$\sigma(\tau, y, M_H^2) = \sum_{ij} \iiint_0^1 dx_1\, dx_2\, dz \int_{-\hat{y}_0}^{\hat{y}_0} d\hat{y} \left[f_i(x_1) f_j(x_2)\delta(y - \xi - \hat{y})\delta(\tau - x_1 x_2 z) C_{ij}(z, \hat{y}, M_H^2)\right]. \tag{4.11}$$

The extremes of integration for the variable $\hat{y}$ are obtained by applying the conditions $M_X^2 \geq 0$ and $|\vec{p}_T|^2 \geq 0$ to the conservation of energy of the four-momenta (4.5), (4.6), (4.7), which leads to

$$\frac{1}{\sqrt{z}} \geq \cosh\hat{y} + |\sinh\hat{y}|. \tag{4.12}$$

From the above inequality, we can derive that the domain of integration with respect to the variable $\hat{y}$ is

$$-\hat{y}_0 = -\frac{1}{2}\ln\left(\frac{1}{z}\right) \leq \hat{y} \leq \frac{1}{2}\ln\left(\frac{1}{z}\right) = \hat{y}_0. \tag{4.13}$$

We now take the Fourier transform with respect to the variable Y and the Mellin transform with respect to τ of both the lhs and rhs of (4.11) and, by defining

$$\sigma(N, M, M_H^2) := \int_0^1 d\tau\, \tau^{N-1} \int_{-y_0}^{y_0} dy\, e^{iMy}\sigma(\tau, y, M_H^2) \tag{4.14}$$

$$C_{ij}(N, M, M_H^2) := \int_0^1 dz\, z^{N-1} \int_{-\hat{y}_0}^{\hat{y}_0} d\hat{y}\, e^{iM\hat{y}} C_{ij}(z, \hat{y}, M_H^2) \tag{4.15}$$

$$f_i\left(N + i\frac{M}{2}\right) := \int_0^1 dx\, x^{N+i\frac{M}{2}-1} f_i(x), \tag{4.16}$$

we obtain the following factored relation:

$$\sigma(N, M, M_H^2) = f_i\left(N + i\frac{M}{2}\right) f_j\left(N - i\frac{M}{2}\right) C_{ij}(N, M, M_H^2). \tag{4.17}$$

4.2.2 From Mellin-Fourier space to Mellin-Mellin space

In this section we describe the change of integration variables $(z, \hat{y}) \mapsto (z_1, z_2)$ which enables us to map the Mellin-Fourier transform into a Mellin-Mellin transform. Once we derive the rapidity distribution factorization, we discuss the constraints which the Mellin variables must satisfy.

It is then convenient to make the following change of variables in definition (4.15):

$$\begin{cases} z_1 = \sqrt{z}e^{\hat{y}} \\ z_2 = \sqrt{z}e^{-\hat{y}} \end{cases} \qquad \begin{cases} z = z_1 z_2 \\ \hat{y} = \frac{1}{2}\ln\left(\frac{z_1}{z_2}\right), \end{cases} \tag{4.18}$$

where we also reported the inverse relations. It is straightforward to show that the Jacobian of this variables transformation is equal to 1 and that the extremes of integration become

$$z \in [0,1]\,,\ \hat{y} \in [-\hat{y}_0(z), \hat{y}_0(z)] \longrightarrow z_1 \in [0,1]\,,\ z_2 \in [0,1]. \tag{4.19}$$

Therefore the definition (4.15) becomes

$$C_{ij}(N, M, M_H^2) = \iint_0^1 dz_1\, dz_2\, z_1^{N+i\frac{M}{2}-1} z_2^{N-i\frac{M}{2}-1} C_{ij}(z(z_1,z_2), \hat{y}(z_1,z_2), M_H^2). \tag{4.20}$$

It is evident that, by calling

$$N_1 \coloneqq N + i\frac{M}{2} \tag{4.21}$$

$$N_2 \coloneqq N - i\frac{M}{2}, \tag{4.22}$$

the relation (4.20) above assumes the more appealing form:

$$C_{ij}(N_1, N_2, M_H^2) = \iint_0^1 dz_1\, dz_2\, z_1^{N_1-1} z_2^{N_2-1} C_{ij}(z(z_1,z_2), \hat{y}(z_1,z_2), M_H^2), \tag{4.23}$$

where $z(z_1, z_2)$ and $\hat{y}(z_1, z_2)$ are the relations (4.18). Moreover, the factored parton model formula (4.17) becomes

$$\sigma(N_1, N_2, M_H^2) = f_i(N_1) f_j(N_2) C_{ij}(N_1, N_2, M_H^2). \tag{4.24}$$

It is now clear the meaning of the change of variables (4.18): we are switching from Mellin-Fourier space to Mellin-Mellin space. The main convenience of this switch comes from the factorization of the domain of integration of z and $\hat{y}$ into the 2 independent domains of z_1 and z_2 reported in (4.19). In fact, in the next section we will show that the coefficient function can be expressed in terms of plus distributions of the variables z_1 and z_2 (example (4.43)) and that the Mellin transforms of plus distributions can be calculated in the large-N limit (see (3.15)).

It is important to make a clarification in order to avoid confusion. In Mellin-Fourier space, since $N \in \mathbb{C}$ and $M \in \mathbb{R}$, we have 3 degrees of freedom, whilst, at first sight, it might seem that in Mellin-Mellin space we have 4 degrees of freedom, because both $N_1 \in \mathbb{C}$ and $N_2 \in \mathbb{C}$. Actually, N_1 and N_2 are defined in (4.21) and (4.22) in terms of N and M, therefore, globally, there are still only 3 degrees of

freedom, as it should be. The missing degree of freedom is due to the fact that N_1 and N_2 have the same real part, because, by expliciting the relations (4.21) and (4.22), we obtain:

$$N_1 = \mathrm{Re}(N) + i[\mathrm{Im}(N) + M/2] \tag{4.25}$$

$$N_2 = \mathrm{Re}(N) + i[\mathrm{Im}(N) - M/2]. \tag{4.26}$$

This fact should be kept in mind in what follows.

4.3 Coefficient functions

In this section, we describe some phenomenology of rapidity distribution computations. Firstly we report two examples of NLO computations of rapidity distributions in terms of z and u. These are not the kinematic variables we chose in sections 4.1 and 4.2, because we used z and $\hat{y}$. Therefore, we describe the variable u, its relation with z and $\hat{y}$, and with z_1 and z_2. Then we explain why u plus distributions are not convenient and how they can be mapped into more convenient z_1 and z_2 plus distributions. Finally, we provide a brief argument which explains how the forward-backward symmetry of the partonic cross section is mapped in Mellin-Fourier and Mellin-Mellin space.

4.3.1 NLO Drell-Yan rapidity distribution

Here we report the partonic rapidity distribution for Drell-Yan process up to NLO as it is presented in [18] adapted to our kinematic notation. The structure is the following:

$$(1-z)\frac{dC_{ij}}{dY} = \eta_{ij}^{(0)} + \frac{\alpha_s}{\pi}\eta_{ij}^{(1)} + O(\alpha_s^3). \tag{4.27}$$

The coefficients are:

$$\frac{\eta_{ij}^{(0)}}{Q_q^2} = (\delta_{iq}\delta_{\bar{q}j} + \delta_{i\bar{q}}\delta_{qj})\delta(1-z)[\delta(u) + \delta(1-u)], \tag{4.28}$$

$$\begin{aligned}\frac{\eta_{q\bar{q}}^{(1)}}{Q_q^2} = \frac{8}{3}\frac{z^2}{1+z}\Bigg\{&[\delta(u) + \delta(1-u)] \\ &\left[\delta(1-z)(2\zeta_2 - 4) + 4\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2(1+z)\ln(1-z) - \frac{1+z^2}{1-z}\ln z + 1 - z\right] \\ &+ \left(1 + \frac{(1-z)^2}{z}u(1-u)\right)\left[\frac{1+z^2}{[1-z]_+}\left(\frac{1}{u_+} + \frac{1}{[1-u]_+}\right) - 2(1-z)\right]\Bigg\}\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}\frac{\eta_{qg}^{(1)}}{Q_q^2} = \frac{z^2}{1+z}\Bigg\{&\delta(u)\left[[z^2 + (1-z)^2]\ln\frac{(1-z)^2}{z} + 2x(1-z)\right] \\ &+ \left(1 + \frac{(1-z)^2}{z}u(1-u)\right)\left[[z^2 + (1-z)^2]\frac{1}{u_+} + 2z(1-z) + (1-z)^2 u\right].\end{aligned} \tag{4.30}$$

4.3.2 NLO Higgs production process rapidity distribution

Here we report the partonic rapidity distribution for Higgs production process up to NLO as it is presented in [19]. Again, we adapted the result to our kinematic notation. The partonic rapidity distribution has been obtained in the large top quark approximation. The structure is the following:

$$\frac{dC_{ij}^h}{dY} = \frac{(1+z)}{2(1-z)\cosh^2(\hat{y})}\sigma_0^h \frac{dC_{ij}^h}{du}, \tag{4.31}$$

where the index $h = H, A$ stands for the CP-even and CP-odd Langrangians. σ_0^h denotes the following expressions:

$$\sigma_0^H = \frac{\pi}{576x^2}\left(\frac{\alpha_s}{\pi}\right)^2 \qquad \sigma_0^A = \frac{9}{4\tan^2\beta}\sigma_0^H, \tag{4.32}$$

where $v \approx 246GeV$ is the void expectation value of the Higgs field, and β is a parameter of the effective theory. $\frac{dC_{ij}^h}{du}$ can be perturbatively expanded as follows:

$$\frac{dC_{ij}^h}{du} = \omega_{ij}^{h,(0)} + \frac{\alpha_s}{\pi}\omega_{ij}^{h,(1)} + O(\alpha_s^2). \tag{4.33}$$

At LO, the only contribution comes from the gg channel and reads as follows:

$$\omega_{gg}^{H,A,(0)} = \frac{1}{2}\delta(1-z)\delta(u(1-u)). \tag{4.34}$$

The NLO coefficients are:

$$\omega_{q\bar{q}}^{H,A,(1)} = \frac{16}{9}(1-z)^3[u^2 + (1-u)^2], \tag{4.35}$$

$$\begin{aligned}\frac{1}{2}\omega_{qg}^{H,A,(1)} + \frac{1}{2}\omega_{gq}^{H,A,(1)} &= \\ &= -(1-z)^2 - \frac{1}{3}\delta(u(1-u))\left\{[1+(1-z)^2]\ln\left(\frac{z}{(1-z)^2}\right) - z^2\right\} \\ &\quad + \frac{1}{3}[1+(1-z)^2]\left[\frac{1}{u(1-u)}\right]_+,\end{aligned} \tag{4.36}$$

$$\begin{aligned}\omega_{gg}^{H,(1)} = \frac{1}{2}\delta(u(1-u))\Bigg\{&\left(6\zeta_2 + \frac{11}{2}\right)\delta(1-z) + 12\left[\frac{\ln(1-z)}{1-z}\right]_+ \\ &- 6(z^2-z+1)^2\frac{\ln(z)}{1-z} - 12z(z^2-z+2)\ln(1-z)\Bigg\} \\ + 3\left\{\left[\frac{1}{1-z}\right]_+ - z(z^2-z+1)\right\}&\left[\frac{1}{u(1-u)}\right]_+ - 3(1-z)^3[2-u(1-u)]\end{aligned} \tag{4.37}$$

and

$$\omega_{gg}^{A,(1)} = \omega_{gg}^{H,(1)} + \frac{1}{4}\delta(1-z)\delta(u(1-u)). \tag{4.38}$$

4.3.3 Plus distributions with respect to z and u

In this section we describe the physical meaning of the variable u and the necessity of changing from the variables (z, u) to (z_1, z_2) in order to obtain plus distributions simpler to manipulate.

Usually the coefficient function is computed in terms of the variables z and u, where z has already been defined in the previous section and

$$u := \frac{e^{-2\hat{y}} - z}{(1-z)(1+e^{-2\hat{y}})} = \frac{z_2(1-z_1^2)}{(1-z_1 z_2)(z_1+z_2)}. \tag{4.39}$$

In appendix B it is shown that u is nothing but $(1+\cos\theta)/2 = \frac{1}{2}(1+X^z/X^0)$, where θ is the angle between an emitted radiation particle X of (4.7) and the collision axe (in the soft limit $M_X^2 = 0$). The variable u is convenient because, since $\cos\theta$ is ranging in $[0, \pi]$, the u variable is ranging in $[0, 1]$, and the coefficient function can be expressed in terms of plus distributions with respect to u.

In definition (4.15), we need to calculate the Mellin-Fourier transform of contributions similar to the one of sections 4.3.1 and 4.3.2. We have to calculate 2 types of terms: Mellin transforms with respect to z of z dependent plus distributions (whose behaviour is well-known, see (3.15)) and Fourier transform with respect to $\hat{y}$ of u dependent plus distributions. The latter type of terms are not immediate to be understood, and now we show that they can be rewritten in a form similar to the Mellin transform with respect to $t := e^{\hat{y}}$ of t dependent plus distributions.

For instance, we consider the Fourier transform of a term of (4.29):

$$\int_{-\hat{y}_0}^{+\hat{y}_0} d\hat{y}\, e^{iM\hat{y}} \frac{1}{u_+} = \tag{4.40}$$

and we substitute the convenient form of u from (B.3), obtaining

$$= \int_{-\hat{y}_0}^{+\hat{y}_0} d\hat{y}\, (e^{iM\hat{y}} - 1) \frac{2(1-z)}{(1-z) - (1+z)\tanh\hat{y}}. \tag{4.41}$$

We retain only the $\hat{y}$ dependent factors, because, for example the $(1-z)$ factor is not to be integrated and it comes out of the integral without affecting its behaviour. We make the above mentioned change of variable $\hat{y} \mapsto t$ and we get

$$= \int_{\sqrt{z}}^{1/\sqrt{z}} dt\, (t^{iM-1} - 1) \frac{2(t^2+1)}{(1-z)(t^2+1) - (1+z)(t^2-1)} = \int_{\sqrt{z}}^{1/\sqrt{z}} dt\, (t^{iM} - 1) \frac{t^2+1}{1-zt^2}. \tag{4.42}$$

It is then clear why the $\hat{y}$ variable is not convenient. In fact we have almost obtained the Mellin transform of a plus distribution, but with the following differences:

- the domain of integration is not $[0, 1]$, therefore we can not apply the usual techniques.
- the denominator $1 - zt^2$ has a pole in $t = 1$ only in the limit $z \to 1$, therefore we should take into consideration the z dependence, too.

4.3.4 Plus distributions with respect to z_1 and z_2

The answer to the problems presented in the previous section is represented by the z_1 and z_2 variables. Firstly, as already pointed out in (4.19), the domains of integration are decoupled and are equal to $[0, 1]$. We now discuss how to manipulate expressions like the ones of sections 4.3.1 and 4.3.2, in order to obtain only Mellin transforms of plus distributions. In (4.23), we need to explicit the variables z_1 and z_2 in terms of z and $\hat{y}$, and it is possible to map plus distributions with respect to z and u in plus distributions with respect to z_1 and z_2. Many of these relations are computed in appendix B of [22] and here we report one example:

$$dz\, d\hat{y}\, \frac{1}{[1-z]_+}\left[\frac{1}{u_+}+\frac{1}{[1-u]_+}\right] = dz_1\, dz_2 \left\{\zeta_2\delta(1-z_1)\delta(1-z_2) - \left[\frac{\ln(1-z_1)}{1-z_1}\right]_+ \right.$$
$$\delta(1-z_2) + \frac{1}{[1-z_1]_+}\frac{1}{[1-z_2]_+} - \delta(1-z_1)\left[\frac{\ln(1-z_2)}{1-z_2}\right]_+$$
$$\left.\delta(1-z_2)\frac{1}{1-z_2}\ln\left(\frac{2z_2}{1+z_2}\right) + \delta(1-z_2)\frac{1}{1-z_1}\ln\left(\frac{2z_1}{1+z_1}\right) + \frac{1}{(1+z_1)(1+z_2)}\right\}, \tag{4.43}$$

where the plus distributions with respect to z_1 and z_2 are intended as plus distributions with respect to the z_1 and z_2 integration separately. To clarify, for instance the following quantity is intended as

$$\iint_0^1 dz_1\, dz_2\, z_1^{N_1-1} z_2^{N_2-1} \frac{1}{[1-z_1]_+}\frac{1}{[1-z_2]_+} = \left[\int_0^1 dz_1 \frac{z_1^{N_1-1}-1}{1-z_1}\right]\left[\int_0^1 dz_2 \frac{z_2^{N_2-1}-1}{1-z_2}\right] \tag{4.44}$$

4.3.5 Forward-backward symmetry

The problem is completely symmetric under interchange of the hadron coming from the right with the hadron coming from the left along the third axe, which is parity with respect to the third axe. This has 3 main consequences on: the coefficient function, its Mellin-Fourier transform and its Mellin-Mellin transform. Firstly, the parity with respect to the third axe implies that the hadronic cross section σ of (4.8) must be symmetric under the interchange $Y \leftrightarrow -Y$, namely

$$\sigma(\tau, Y, M_H^2) = \sigma(\tau, -Y, M_H^2). \tag{4.45}$$

On the one hand, if the 2 extracted partons are identical, i.e. $i = j$, the coefficient function C_{ii} is symmetric under the interchange $\hat{y} \leftrightarrow -\hat{y}$, too, because also the partonic problem is symmetric under parity with respect to the third axe. On the other hand, if $i \neq j$, the coefficient function is not symmetric under $\hat{y} \leftrightarrow -\hat{y}$. However, since in the hadronic cross section we are summing over all possible extracted partons, we can redefine $\tilde{C}_{ij} := C_{ij} + C_{ji}$, which now is symmetric under $\hat{y} \leftrightarrow -\hat{y}$, too. With a little abuse of notation, from now on C_{ij} will be intended as $\tilde{C}_{ij}$, and the some over all possible extracted partons will be intended as $\sum_{i>j}$.

The forward-backward symmetry of the coefficient function implies that its Mellin-Fourier transform is symmetric under $M \leftrightarrow -M$. In fact, if we consider (4.15) and we make the change of variable $\hat{y} \mapsto -\hat{y}$, we get:

$$C_{ij}(N, M, M_H^2) = \int_0^1 dz\, z^{N-1} \int_{-\hat{y}_0}^{+\hat{y}_0} d\hat{y}\, e^{-iM\hat{y}} C_{ij}(z, \hat{y}, M_H^2) = C_{ij}(N, -M, M_H^2). \tag{4.46}$$

Moreover, if we consider the definitions (4.21) and (4.22), it is straightforward to see that the transformation $M \leftrightarrow -M$ corresponds to $N_1 \leftrightarrow N_2$. Therefore the following relation must hold, too:

$$C_{ij}(N_1, N_2, M_H^2) = C_{ij}(N_2, N_1, M_H^2). \tag{4.47}$$

4.4 Thresholds

In this section we analyse the kinematic threshold of the considered process. Firstly, we describe the doubly soft case, starting from the $(z, \hat{y})$ limits and obtaining the corresponding limits in Mellin-Fourier space, i.e. (N, M) variables. Then we describe to what limits the doubly soft region corresponds in (z_1, z_2) variables and in Mellin-Mellin space, i.e. (N_1, N_2) variables.

Secondly, we move to the singly soft case. We start by deriving the (z_1, z_2) limits in the singly soft region and then we derive the corresponding (N_1, N_2) limits. Finally, we switch to $(z, \hat{y})$ variables and we describe their limits in the singly soft region and what is the corresponding behaviour in Mellin-Fourier space, i.e. (N, M) variables.

4.4.1 Doubly soft limit

In this section we derive that the doubly soft region corresponds to $(z \to 1, \hat{y} \to 0)$ and $(|N| \to \infty, M \to \pm\infty)$, where $N = \rho M$ and ρ is a finite complex parameter.

Then we derive that the doubly soft region corresponds to $(z_1 \to 1, z_2 \to 1)$ and $(|N_1| \to \infty, |N_2| \to \infty)$, where $\frac{N_1}{N_2} = \frac{\rho + i/2}{\rho - i/2}$ is finite.

Doubly soft in $(z, \hat{y})$ variables and (N, M) Mellin-Fourier variables

Here we analyse the doubly soft region[1], where the available energy $\hat{s}$ is approaching its minimum allowed value in order to produce a real Higgs, which is M_H^2. Therefore, by definition, $\hat{s} \to M_H^2$ implies $z \to 1$. It is important to keep in mind that, for $z \to 1$, the extremes of integration $\pm\hat{y}_0$ of (4.13) in expression (4.15) are approaching $0^{\pm}$ because

$$\lim_{z \to 1} \hat{y}_0 = \lim_{z \to 1} \frac{1}{2} \ln\left(\frac{1}{z}\right) = 0, \tag{4.48}$$

[1]the name "doubly soft" limit will be clear in section 4.4.1, where it is shown that $z \to 1$ corresponds to $z_1 \to 1$ and $z_2 \to 1$, while the so-called "singly soft" limit, discussed in section 4.4.2, means taking the z_1 and z_2 one at a time.

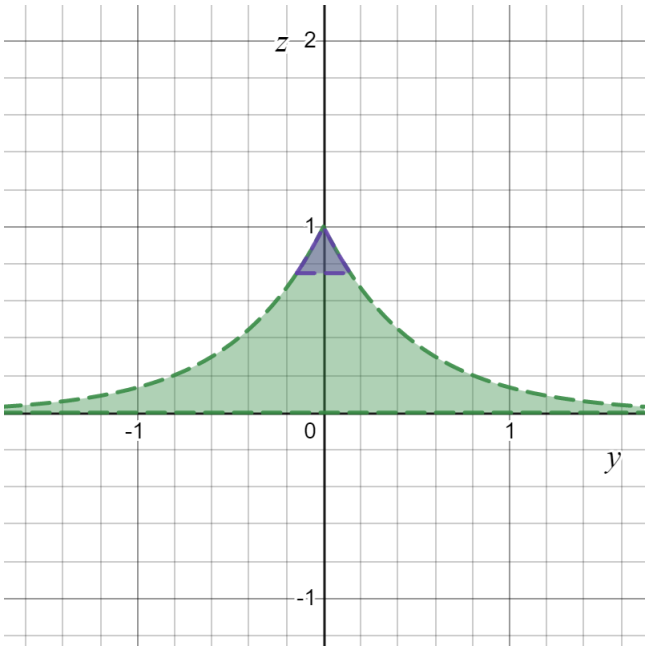


Figure 4.1: Domain of integration with respect to the variables z and $\hat{y}$ and the doubly soft region

therefore the set of allowed values of $\hat{y}$ is becoming a narrow range centred around 0. Figure 4.1 represents the domain of integration and it clears why taking the soft limit $z \to 1$ implies $\hat{y} \to 0$. The physical meaning of this fact is rather obvious: if the available energy is approaching its minimum value, then there is no more left for the longitudinal momentum of the Higgs.

Now we need to understand to which (N, M) limits the doubly soft region corresponds. Firstly, we want to show that, by taking the limit $|N| \to \infty$ of Mellin transform of the coefficient function, we are restricting ourselves to the $z \to 1$ region. To show that, we starting by considering the definition (4.15)

$$C_{ij}(N, M, M_H^2) = \int_0^1 dz\, z^{N-1} \int_{-\hat{y}_0}^{+\hat{y}_0} d\hat{y}\, e^{iM\hat{y}} C_{ij}(z, \hat{y}, M_H^2) = \int_0^1 dz\, z^{N-1} C_{ij}(z, M, M_H^2) \tag{4.49}$$

and we focus on the z^{N-1} factor. In fact, since $z \in [0, 1]$, by taking the $|N| \to \infty$ implies that $z^{N-1} \to 0$ unless $z \to 1$. A similar argument can be applied to the inverse Mellin transform to show that also $z \to 1$ implies $|N| \to \infty$. This mapping can be expressed more quantitatively by calculating explicitly the Mellin transform of a typical expression of the rapidity distribution in the large-N limit as the example (3.15), where only a finite number of terms provide non-suppressed contributions.

Secondly, we show that $M \to \pm\infty$ corresponds to the $\hat{y} \to 0$ region. In fact, for $M \to \pm\infty$, the $e^{iM\hat{y}}$ factor is a function, which is oscillating increasingly fast, and the only way to obtain a finite factor $e^{iM\hat{y}}$ is for $\hat{y} \to 0$. A similar argument applied to the inverse Fourier transform shows that $\hat{y} \to 0 \leftrightarrow M \to \pm\infty$. Since in section 4.3.5 we showed that the Mellin-Fourier transform of the coefficient function is symmetric under parity with respect to the M variable, from now on we will drop

the $M \to \pm\infty$ distinction, because these 2 limits are indistinguishable.

When considering terms of the form (4.42), obtaining a quantitative relation for this fact is a hard task, because one can not consider the Fourier transform alone, but has to consider it inside a Mellin transform. Here we give a specific example to get a good grasp on the emergence of big logs of M and, to do that, we consider the following contribution to the Mellin-Fourier transform of the coefficient function:

$$\int_0^1 dz\, z^{N-1} \frac{1}{[1-z]_+} \int_{-\hat{y}_0}^{+\hat{y}_0} d\hat{y}\, e^{iM\hat{y}} \frac{1}{u_+}. \tag{4.50}$$

If we make the same change of variable as in 4.3.3 and we isolate one of the many singular contributions, we get

$$\approx \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \int_{\sqrt{z}}^{1/\sqrt{z}} dt\, \frac{t^{iM}-1}{1/\sqrt{z}-t} + \dots\,. \tag{4.51}$$

Then, since both the extremes of integration are approaching 1, we can approximate the above quantity as

$$\approx \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \frac{1-z}{\sqrt{z}} \frac{\sqrt{z}^{iM}-1}{\frac{1-z}{\sqrt{z}}} \approx \int_0^1 dz \frac{z^{N+iM/2-1}-1}{1-z} = -\ln(N+iM/2) + O(1), \tag{4.52}$$

where in the last step we dropped some suppressed terms and we used the relation (3.15) with $p = 0$. In (4.52) it can be seen how logs of M do not appear immediately after the Fourier transform and independently of logs of N, but they are built by the Mellin transform with the outcome of the Fourier transform.

Lastly, we focus on the relative rates of the $|N| \to \infty$ and $M \to \infty$ limits. We consider again the definition (4.15), where, for $z \to 1$, we can safely say that the $\hat{y}$ variable in the inner integral is approaching 0 at most at the same rate as its extremes of integration $|\hat{y}_0| = \frac{1}{2}\ln\frac{1}{z}$, hence as a first approximation $e^{iM\hat{y}} \approx z^{iM/2}$ and (4.15) becomes

$$C_{ij}(N, M, M_H^2) \approx \int_0^1 dz\, z^{N-1} \int_{-\hat{y}_0}^{\hat{y}_0} d\hat{y}\, z^{i\frac{|M|}{2}} C_{ij}(z, \hat{y}, M_H^2). \tag{4.53}$$

Therefore, if $|N|$ is required to approach ∞ at a certain rate, then, in order to have a non-trivial factor $z^{i\frac{M}{2}}$, M must approach ∞ at the same rate of N, i.e. $M = O(|N|)$. We are left with the freedom of choosing the complex numeric value

$$\rho := \frac{N}{M} = \frac{|N|}{M} e^{i\phi}, \tag{4.54}$$

where we decomposed $N = |N|e^{i\phi}$, the modulus of ρ is determined by $|\rho| = |N|/M$ and it represents the relative rate of the limits, and its phase is determined by $\rho/|\rho| = N/|N|$. The complex number ρ can be thought as the parametrization of the doubly soft region in Mellin-Fourier space.

Doubly soft in (z_1, z_2) variables and (N_1, N_2) Mellin variables

We now focus on expression (4.23) and we ask to what this doubly soft limit corresponds with respect to the variables $z_1 = \sqrt{z}e^{\hat{y}}$ and $z_2 = \sqrt{z}e^{-\hat{y}}$. Taking the limit $z \to 1$ implies that $z_1 \to e^{\hat{y}}$ and $z_2 \to e^{-\hat{y}}$, and moreover, since both the maximum and the minimum allowed values for $\hat{y}$ are approaching 0, then $z_1 \to 1$ and $z_2 \to 1$ at the same time (hence the name "doubly soft").

We now focus on the Mellin-Mellin limits to which this (z_1, z_2) region corresponds. As already discussed in the previous section, the $z \to 1$ region corresponds to the $|N| \to \infty$ limit. In Mellin-Mellin space the situation is not different, because we have 2 independent Mellin transforms with respect to z_1 and z_2, which are both approaching 1, therefore we know that $|N_1| \to \infty$ and $|N_2| \to \infty$. Up to this point, we are still free to take the 2 limits at different rates, but we need to do a more careful analysis. We have already done some considerations about the rates of the limits $M \to \infty$ and $|N| \to \infty$ and we came to the relation $M = O(|N|)$. Due to the relations $N_1 = N + iM/2$ and $N_2 = N - iM/2$, we are obviously constrained to the situation where $|N_1| = O(|N_2|) = O(|N|) = O(M)$ in the doubly soft limit. The freedom we are left with is the same as in Mellin-Fourier space, which is the complex parameter ρ. Now ρ parametrize the limits $|N_1| \to \infty$ and $|N_2| \to \infty$ as expressed by

$$N_1 = M(\rho + i/2) \tag{4.55}$$

$$N_2 = M(\rho - i/2) \tag{4.56}$$

Figure 4.2 represents the values M, N, N_1 and N_2 in the complex plane in units of M, in order to consider only the relative rates of the limits.

Reversing the argument of the previous paragraph, we can deduce that, since $|N_1| = O(|N_2|)$, the regions in (z_1, z_2) space are $1 - z_1 = O(1 - z_2)$, which is the analogue of the request for $\hat{y}$ to approach 0 at rate similar to its extremes of integration. Figure 4.3 is explanatory of the described situation, where (z_1, z_2) are approaching the top right corner along every possible line, because they are of the same order.

4.4.2 Singly soft limit

In this section we derive that the first singly soft region corresponds to (maximum rapidity) $(z\text{fixed}, \hat{y} \to \hat{y}_0)$ and $(|N| \to \infty, M \to \pm\infty)$, where $\mathrm{Re}(N)$ is finite and $\mathrm{Im}(N) = M/2$. The other singly soft region corresponds to (minimum rapidity) $(z\text{fixed}, \hat{y} \to -\hat{y}_0)$ and $(|N| \to \infty, M \to \pm\infty)$, where $\mathrm{Re}(N)$ is finite and $\mathrm{Im}(N) = -M/2$.

Then we derive that the first singly soft region corresponds to $(z_1 \to 1, z_2\text{fixed})$ and $(|N_1| \to \infty, |N_2|\text{fixed})$, where $\mathrm{Re}(N_1)$ is finite. The second singly soft region can be obtained by interchanging $1 \leftrightarrow 2$.

Singly soft in (z_1, z_2) and (N_1, N_2) Mellin variables

In the kinematic configuration considered in the previous section, since all the available energy is used to produce the Higgs, the extra radiation is becoming soft and resummation is needed. We now deal with a different kinematic configuration,

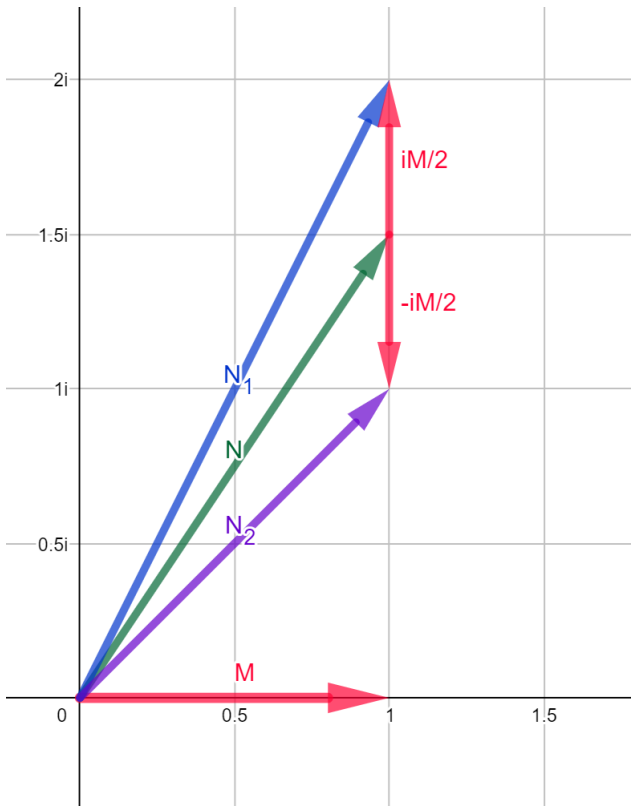


Figure 4.2: The limits $M, |N|, |N_1|, |N_2| \to \infty$ are represented in the complex plane in units of M, and relative rates are obtained from the relations $N = M\rho$, $N_1 = M(\rho + i/2)$ and $N_2 = M(\rho - i/2)$.

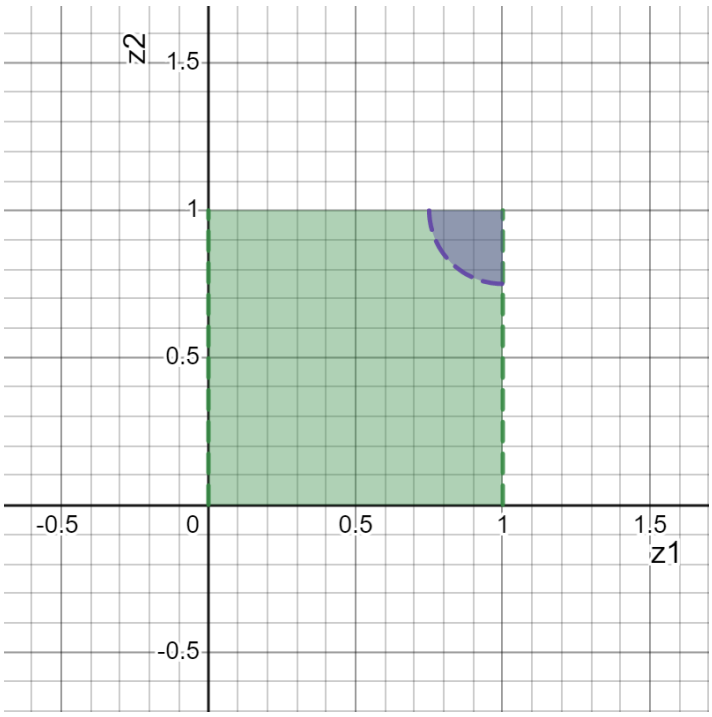


Figure 4.3: Domain of integration with respect to the variables z_1 and z_2 and doubly soft region

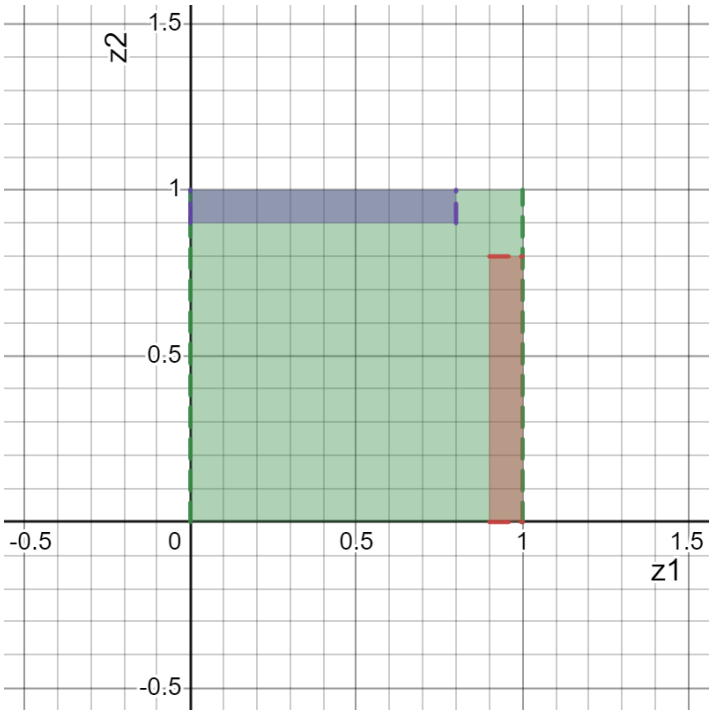


Figure 4.4: Domain of integration with respect to the variables z_1 and z_2 and singly soft regions

where z is not approaching 1, so there is extra energy available for the Higgs, but the Higgs is using as much extra energy as possible in order to have the greatest/smallest possible longitudinal momentum, therefore its transverse momentum is approaching 0. In this configuration, the extra radiation allowed is of 2 types: collinear to the Higgs, in order to compensate the longitudinal momentum of the Higgs because of the conservation of the four-momentum, and the remaining radiation has to become soft, therefore resummation is needed also in this case.

We start from relation (4.23) and we consider the case $\hat{y} \to \hat{y}_0$, because the case $\hat{y} \to -\hat{y}_0$ is symmetric. We can immediately read why using the variables z_1 and z_2 is easier. Since the variables can be rewritten in the form $z_1 = e^{\hat{y}-\hat{y}_0}$ and $z_2 = e^{-\hat{y}-\hat{y}_0}$, the limits above correspond to $z_2 < 1$ fixed and $z_1 \to 1$, and $z_1 < 1$ fixed and $z_2 \to 1$. In these limits, respectively $z \to z_2$ or $z \to z_1$, therefore it is not approaching 1. Figure 4.4 is explanatory of the kinematic regions we are considering in these 2 limits in (z_1, z_2) space.

Now we show to which limits of the Mellin variables (N_1, N_2) the regions $z_1 \to 1$ and $z_2 \to 1$ correspond. In analogy with the previous section, we consider (4.23)

$$C_{ij}(N_1, N_2, M_H^2) = \int_0^1 dz_1\, z_1^{N_1-1} C_{ij}(z_1, N_2, M_H^2), \tag{4.57}$$

where we focus on the z_1 integration. For $|N_1| \to \infty$, then the factor $z_1^{N_1-1} \to 0$, unless $z_1 \to 1$. On the other hand, the variable z_2 is freely ranging in its domain, therefore it is not necessary to take any limit for N_2. The case in which z_1 and z_2 are playing each other's role is completely analogous, with the only needed substitution $1 \leftrightarrow 2$.

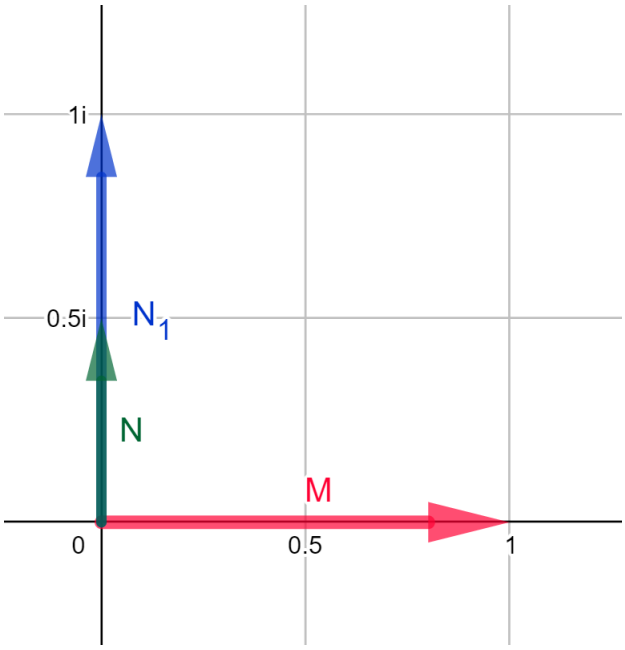


Figure 4.5: The limits $M, |N|, |N_1| \to \infty$ are represented in the complex plane in units of M. Since N_2 is finite, it can not be represented because it corresponds to 0 in M units. The relative rates are obtained from the relations $N = iM/2$, $N_1 = M(\rho + i/2) = iM$.

Since we are taking only 1 limit, there is no relative rate between N_1 and N_2 to reflect on, but, since N_1 and N_2 are not 2 independent complex numbers, there are some important consequences when asking $|N_1| \to \infty$ and N_2 finite. In fact, by looking at (4.25) and (4.26), the fact that N_1 and N_2 have the same real part implies that the real part of both of them must be finite. Therefore N_1 must approach ∞ along the imaginary axe, neglecting sub-leading contributions. The explained situation is depicted in figure 4.5.

Singly soft in $(z, \hat{y})$ and (N, M) Mellin-Fourier variables

If we analyse the same situation with respect to the variables $(z, \hat{y})$, it is straightforward to understand to what these kinematic regions correspond. In fact, z is fixed to a numeric value, and is not approaching 1, while the variable $\hat{y}$ is approaching $\pm\hat{y}_0$, depending on whether we are in the $z_1 \to 1$ or in the $z_2 \to 1$ case, keeping in mind that $\hat{y}_0(z)$ is z dependent. The described situation is depicted in figure 4.6.

Lastly, we explain to which limits these regions correspond in Mellin-Fourier space. In order to do this, we exploits again the relations $N_1 = N + iM/2$ and $N_2 = N - iM/2$ and we ask ourselves what they tell us about N and M if $N_1 \to \infty$ and N_2 is finite. By looking at (4.25) and (4.26) we can deduce what follows. The difference $N_1 - N_2$ of (4.25) and (4.26) must be infinite because $|N_1| \to \infty$ and N_2 is finite, hence $M \to \infty$. On the other hand, N_2 must be finite, therefore both its real and imaginary parts must be finite, hence $\mathrm{Re}(N)$ is finite and $\mathrm{Im}(N) \to \infty$ in order to compensate the divergence of M, and in particular $\mathrm{Im}(N) \cong M/2$ up to a constant. Moreover, since N is becoming increasingly imaginary, we get

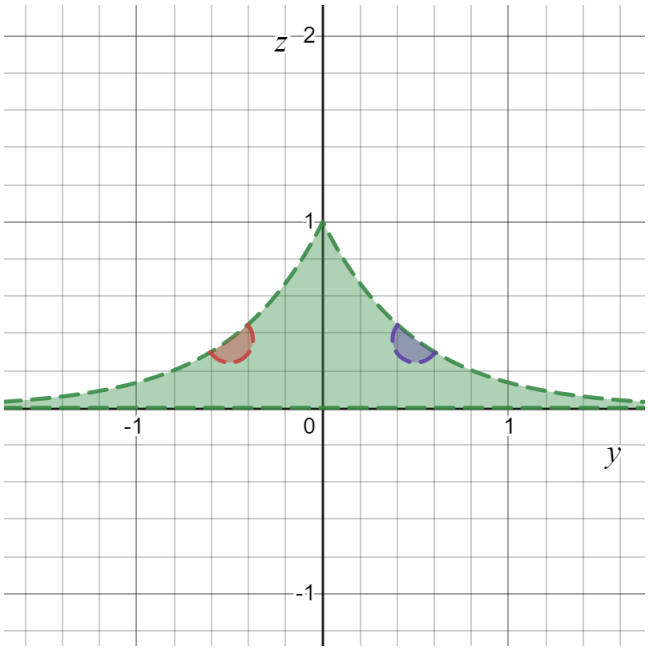


Figure 4.6: Domain of integration with respect to the variables z and $\hat{y}$ and the singly soft regions

$N \to i\,\mathrm{Im}(N)$ and, by recalling the definition $\rho = N/M$, we obtain the following value for ρ:

$$\rho \to \frac{i\,\mathrm{Im}(N)}{M} \to \frac{i}{2}. \tag{4.58}$$

This is exactly the expected result because, by dividing (4.55) by (4.56), we get

$$\frac{N_1}{N_2} = \frac{\rho + i/2}{\rho - i/2}, \tag{4.59}$$

which is divergent for $\rho \to i/2$. In figure 4.5, the explained situation is depicted, and one can see that this is a particular case of figure 4.2, with the substitution $\rho = i/2$. The symmetric result can be found for N_1 finite $N_2 \to \infty$, which, at the end, requires $\rho \to -i/2$.

4.5 Phase Space

In this section we analyse the phase space structure of the radiation emission.

Firstly, we reorganise the phase space integration measure by dividing it into soft radiation and radiation collinear to the massive final state particle.

Secondly, we analyse the singly soft limit and we show that the phase space can depend only on two combination of variables: the collinear scale $M_H^2(1-z_1)(1-z_2)$ and the soft scale $M_H^2(1-z_1)^2$, where z_2 is fixed and $z_1 \to 1$.

Thirdly, we analyse the doubly soft limit and we show that the phase space depends on the same scales of the singly soft limit, but this time we are taking both $z_1 \to 1$ and $z_2 \to 1$. We discuss the meaning of this result and its interpretation.

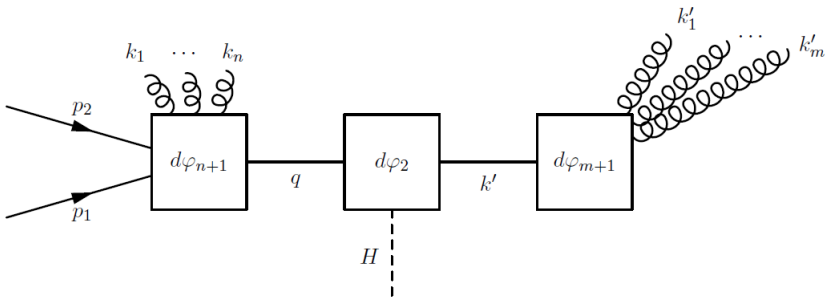


Figure 4.7: Phase space decomposition of the process with two incoming particles with 4-momenta p_1 and p_2, a final state massive particle H, m soft gluons with 4-momenta $k_1, ...k_m$ and m gluons collinear to H and with 4-momenta $k'_1, ..., k'_n$.

4.5.1 Phase space for rapidity distributions

The following treatment follows a procedure similar to [15]. In this section we manipulate the phase space in order to factor soft and collinear radiation. We distinguish between two classes of extra radiation: soft radiation and radiation collinear to the massive particle H. Therefore the conservation of the four-momenta for the process (4.3) takes the form

$$p_1 + p_2 = p_H + k_1 + \ldots + k_m + k'_1 + \ldots + k'_n, \tag{4.60}$$

where we denoted with k_i a soft radiation particle and with k'_i a collinear radiation particle. The infinitesimal element of the phase space in $d = 4 - 2\epsilon$ dimensions takes the following form:

$$\begin{aligned} &d\phi_{m+n+1}(p_1, p_2; p_H, k_1, \ldots, k'_n) = \\ &= \frac{d^{d-1}p_H}{(2\pi)^{d-1}2p_H^0}\frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0}\cdots\frac{d^{d-1}k'_n}{(2\pi)^{d-1}2k_n'^0}(2\pi)^d\delta^{(d)}(p_1 + p_2 - p_H - k_1 - \ldots - k'_n). \end{aligned} \tag{4.61}$$

The phase space can be divided into sub-phase spaces by introducing intermediate particles and integrating over their four-momenta squared. Our choice is to split the phase space the way it has been done in [15] for transverse momentum distributions, which reads:

$$= \int \frac{dq^2}{2\pi}\, d\phi_{m+1}(p_1, p_2; q, k_1, \ldots, k_m) \int \frac{d(k')^2}{2\pi}\, d\phi_2(q; p_H, k')d\phi_n(k'; k'_1, \ldots, k'_n). \tag{4.62}$$

Figure 4.7 illustrates the chosen phase space decomposition.

Now, we calculate explicitly the element $d\phi_2(q; p_H, k')$, keeping in mind that we need to obtain a rapidity distribution, therefore we have to isolate the measure dp_H^z. By definition, we know that

$$d\phi_2(q; p_H, k') = \frac{d^{d-1}p_H}{(2\pi)^{d-1}2p_H^0}\frac{d^{d-1}k'}{(2\pi)^{d-1}2k'^0}(2\pi)^d\delta^{(d)}(q - p_H - k'), \tag{4.63}$$

we make the spatial $\delta^{(d-1)}$ act to cancel the integration measure $d^{d-1}k'$, we choose the frame of reference in which q is at rest and we are left with

$$= \frac{d^{d-1}p_H}{4(2\pi)^{d-2}p_H^0 k'^0}\delta^{(1)}(\sqrt{q^2} - p_H^0 - k'^0). \tag{4.64}$$

The integration measure can be rewritten as

$$d^{d-1}p_H = d^2\vec{p}_T\, d^{d-3}\vec{p}_z = \frac{1}{2}dp_T^2\, |\vec{p}_z|^{d-4} d|\vec{p}_z|\, d\Omega_{d-2} = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}dp_T^2\, |\vec{p}_z|^{-2\epsilon}\, d|\vec{p}_z|, \tag{4.65}$$

where we exploited the relation $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Substituting it into (4.64), it becomes

$$d\phi_2(q; p_H, k') = \frac{(4\pi)^\epsilon |\vec{p}_z|^{-2\epsilon}}{16\pi\Gamma(1-\epsilon)}\frac{dp_T^2\, d|\vec{p}_z|}{p_H^0 k'^0}\delta^{(1)}(\sqrt{q^2} - p_H^0 - k'^0). \tag{4.66}$$

Then we want to make the $\delta^{(1)}$ act on the integration measure dp_T^2, therefore we the known property of the δ of a function, obtaining

$$\delta(p_H^0 + k'^0 - \sqrt{q^2}) = \frac{\delta(|\vec{p}_T|^2 - \tilde{p}_T^2)}{|J(\tilde{p}_T^2)|}, \tag{4.67}$$

where

$$\tilde{p}_T^2 = \frac{\lambda(M_H^2, q^2, (k')^2)}{4q^2} - p_z^2, \qquad \lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{4.68}$$

and

$$|J(\tilde{p}_T^2)| = \frac{1}{2}\left(\frac{1}{p_H^0} + \frac{1}{k'^0}\right) = \frac{\sqrt{q^2}}{2p_H^0 k'^0} \tag{4.69}$$

Substituting (4.67) and (4.69) into (4.66) we get:

$$d\phi_2(q; p_H, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1/2-\epsilon)}\frac{|\vec{p}_z|^{-2\epsilon}}{\sqrt{q^2}}d|\vec{p}_z|. \tag{4.70}$$

The last step of our manipulation is transforming the $d|\vec{p}_z|$ differential into a $d|\hat{y}|$ rapidity differential, because we are interested in rapidity distributions. The procedure is the following:

$$d|\vec{p}_z| = \sqrt{M_H^2 + p_T^2}\cosh(\hat{y})d|\hat{y}| = \sqrt{M_H^2 + p_T^2}\frac{z_1 + z_2}{2\sqrt{z_1 z_2}}d|\hat{y}| \tag{4.71}$$

and substituting this relation into (4.70), it becomes

$$d\phi_{m+n+1}(p_1, p_2; p_H, k_1, \ldots, k'_n) == \frac{(4\pi)^\epsilon |p_z|^{-2\epsilon}}{32\pi^3\Gamma(1-\epsilon)}\sqrt{M_H^2 + p_T^2}\frac{z_1 + z_2}{2\sqrt{z_1 z_2}}d|\hat{y}|$$
$$\int \frac{dq^2}{\sqrt{q^2}}\int d(k')^2\, d\phi_{n+1}(p_1, p_2; q, k_1, \ldots, k_m)d\phi_m(k'; k'_1, \ldots, k'_m). \tag{4.72}$$

It is important to note that we seem to have lost information about the sign of $\hat{y}$ because we found cross section differential in the modulus of the longitudinal momentum, but actually the information about the sign is stored inside the variable among z_1 and z_2 which is approaching 1 faster, both in the case of single limit and double limit. In fact it is straightforward to prove from (4.18) that if $z_1 > z_2$ the longitudinal momentum has a positive sign, and vice versa for the other case.

4.5.2 Singly soft limit: extremes and scales

In the section analyse the extremes of the phase space integration with respect to soft and collinear radiation in the singly soft case. From them, we deduce the soft and collinear scales.

We aim to understand what are the extremes of integration in (4.72) in the situation when only one of the variables z_1 and z_1 is approaching 1. In this section we assume that $z_1 \to 1$ and the other case is perfectly symmetric up to the substitution $(1 \mapsto 2)$.

We start by writing the conservation of four-momenta for $d\phi_{n+1}$, which is

$$p_1 + p_2 = q + k_1 + \ldots + k_m, \tag{4.73}$$

and, by squaring this equation, it becomes

$$\hat{s} = q^2 + 2q^0 \sum_{i=1}^{m} |\vec{k}_i| - 2\sqrt{(q^0)^2 - q^2} \sum_{i=1}^{m} |\vec{k}_i| \cos\theta_i + 2 \sum_{i,j=1}^{m} |\vec{k}_i||\vec{k}_j|(1 - \cos\theta_{ij}). \tag{4.74}$$

It is clear that the maximum value of q^2 is reached in the limit of soft radiation, i.e. $|\vec{k}_i| \to 0 \ \forall i \Longrightarrow q^2 \to \hat{s}$.

The lower extreme of integration can be found by looking at the conservation of energy for $d\phi_2(q; p_H, k')$ in the frame of reference of q, which reads

$$q^2 = \left(\sqrt{M_H^2 + p_T^2 + p_z^2} + \sqrt{(k')^2 + p_T^2 + p_z^2} \right)^2, \tag{4.75}$$

where we have identified z in the frame of reference of q with the one in the partonic frame of reference because in the soft limit these two frames coincide. Lowering $(k')^2$ and p_T^2 down to 0, we obtain that

$$q^2 \geq q_{min}^2 = \left(\sqrt{M_H^2 + p_z^2} + |p_z| \right)^2. \tag{4.76}$$

We prefer to express this lower limit as a function of z_1 and z_2. Substituting $p_z = M_H \sinh \hat{y}$ because of $p_T^2 = 0$, and knowing (4.18), we get

$$q_{min}^2 = \frac{\hat{s}}{4}(z_1 + z_2 + |z_1 - z_2|)^2 = (\max\{z_1, z_2\})^2 \hat{s} = z_1^2 \hat{s}, \tag{4.77}$$

where $\max\{z_1, z_2\} = z_1$ because we chose the configuration $z_1 \to 1$ and $z_2 \neq 1$. Therefore, the integration interval is $q^2 \in [z_1^2 \hat{s}, \hat{s}]$.

We choose a new dimensionless variable u instead of q^2, defined by interpolating the extremes of integration as follows

$$q^2 = z_1^2 \hat{s} + u(\hat{s} - z_1^2 \hat{s}) = \hat{s}(z_1^2 + u(1 - z_1^2)). \tag{4.78}$$

If we make this change of variable, eq. (4.72) becomes

$$d\phi_{m+n+1}(p_1, p_2, k_1, \ldots, k_n') = \frac{(4\pi)^\epsilon |p_z|^{-2\epsilon}}{32\pi^3 \Gamma(1-\epsilon)} \sqrt{M_H^2 + p_T^2} \frac{z_1 + z_2}{2\sqrt{z_1 z_2}} d|\hat{y}| \sqrt{\hat{s}}(1 - z_1^2)$$
$$\int_0^1 \frac{du}{\sqrt{z_1^2 + u(1 - z_1^2)}} \int d(k')^2 \, d\phi_{n+1}(p_1, p_2; q, k_1, \ldots, k_m) d\phi_m(k'; k_1', \ldots, k_m'). \tag{4.79}$$

Regarding the integration over k', the lower limit is trivially found to be 0, while for the upper one we need to recall (4.75), which leads to

$$(k')^2 = q^2 + M_H^2 - 2\sqrt{q^2}\sqrt{M_H^2 + p_T^2 + p_z^2}. \tag{4.80}$$

Then the maximum value is obtained for $p_T^2 \to 0$, hence it reads as

$$(k'_{max})^2 = q^2 + M_H^2 - 2\sqrt{q^2}\sqrt{M_H^2 + p_z^2}. \tag{4.81}$$

If we isolate a factor $\hat{s}$ and we substitute $q^2/\hat{s}^2 = z_1^2 + u(1 - z_1^2)$, $M_H^2/\hat{s} = z = z_1 z_2$ and $p_z^2/\hat{s}^2 = (z_1 - z_2)^2/4$, it results in

$$= \hat{s}\left(z_1^2 + u(1 - z_1^2) + z_1 z_2 - (z_1 + z_2)\sqrt{z_1^2 + u(1 - z_1^2)}\right), \tag{4.82}$$

which tends to 0 in the limit $z_1 \to 1$. Therefore we expand it to the lowest order in powers of $\eta = 1 - z_1$, obtaining

$$\begin{aligned} &= \hat{s}\,(1 - 2\eta + 2u\eta + z_2 - z_2\eta - (1 + z_2 - \eta)(1 - \eta(1 - u))) + O(\eta) \\ &= \hat{s}u(1 - z_2)\eta + O(\eta) = \hat{s}u(1 - z_1)(1 - z_2) + O\left((1 - z_1)^2\right). \end{aligned} \tag{4.83}$$

It is important to keep in mind that in this case the singular contribution is represented by $1 - z_1$, while $1 - z_2$ is just to be intended as a numeric constant, for the moment.

Once again, we choose an dimensionless variable v in order to interpolate the extremes of integration 0 and $(k'_{max})^2$, which is defined as

$$(k')^2 = uv(1 - z_1)(1 - z_2)\hat{s}. \tag{4.84}$$

Finally, by applying this change of variable to (4.79), we get

$$\begin{aligned} &\frac{(4\pi)^\epsilon |p_z|^{-2\epsilon}}{32\pi^3\Gamma(1-\epsilon)}\sqrt{M_H^2 + p_T^2}\frac{z_1 + z_2}{2\sqrt{z_1 z_2}}d|\hat{y}|\hat{s}^{3/2}(1 - z_1)^2(1 + z_1)(1 - z_2) \\ &\int_0^1 \frac{udu}{\sqrt{z_1^2 + u(1 - z_1^2)}}\int_0^1 dv\, d\phi_{n+1}(p_1, p_2; q, k_1, \ldots, k_m)d\phi_m(k'; k'_1, \ldots, k'_m). \end{aligned} \tag{4.85}$$

The formula above can be simplified by taking the limit $z_1 \to 1$ and $p_T^2 \to 0$ in the non singular terms and by substituting $\hat{s} = \frac{M_H^2}{z_1 z_2} \to \frac{M_H^2}{z_2}$. Then, it becomes:

$$\begin{aligned} &= \frac{(4\pi)^\epsilon |p_z|^{-2\epsilon}}{32\pi^3\Gamma(1-\epsilon)}M_H^4(1 - z_1)^2\frac{(1 - z_2^2)}{z_2^2} \\ &\qquad \int_0^1 udu\int_0^1 dv\, d\phi_{n+1}(p_1, p_2; q, k_1, \ldots, k_m)d\phi_m(k'; k'_1, \ldots, k'_m). \end{aligned} \tag{4.86}$$

The 2 remaining phase spaces can be dealt with as in appendix A of [13]:

- $d\phi_{n+1}$ has the form of a Drell-Yan phase space and it depends on dimension full variables only through the combination

$$\frac{(\hat{s} - q^2)^2}{q^2} = \hat{s}\frac{[1 - (z_1^2 + u(1 - z_1^2))]^2}{z_1^2 + u(1 - z_1^2)} \sim \frac{4(1 - u)^2}{z_2}M_H^2(1 - z_1)^2. \tag{4.87}$$

- $d\phi_m$ has the form of a DIS phase space and it depends on dimension full variables only through the combination

$$(k')^2 = \hat{s}uv(1-z_1)(1-z_2) \sim uvM_H^2(1-z_1)\frac{1-z_2}{z_2}. \tag{4.88}$$

From these dependences we understand that, for $z_1 \to 1$, the phase space is a function of dimension full variables only through the 2 following combinations:

$$\Lambda_{DY}^2 = M_H^2(1-z_1)^2, \tag{4.89}$$

which emerges as a soft scale, and

$$\Lambda_{DIS}^2 = M_H^2(1-z_1)(1-z_2), \tag{4.90}$$

which, on the other hand, emerges as a collinear scale.

4.5.3 Doubly soft limit: extremes and scales

In the section analyse the extremes of the phase space integration with respect to soft and collinear radiation in the doubly soft case. The scales we obtain are identical to the ones of the singly soft limit, but they have different interpretations and behaviours.

In this case we have to find what are the extremes of integration, too. Most of the calculations done in the previous section are still true in this case. In particular, regarding the extremes of integration with respect to the variable q^2 are the same, because they are exact and they did not go through any approximation. It is also true the fact that the lower limit is $z_1^2\hat{s}$ if z_1 is the variable approaching 1 faster. In this case, the results will still be perfectly symmetric under the interchange $z_1 \leftrightarrow z_2$.

Regarding the extremes of integration with respect to $(k')^2$, we should in principle expand the upper limit in powers of $1-z_1$ and $1-z_2$, keeping in mind that these 2 quantities are of the same order. Actually, the calculations done before have already provided us with the result we are looking for, because we approximated only with respect to the variable z_1, therefore the result is exact with respect to z_2.

Therefore, we can resume the discussion starting from (4.85), which becomes

$$\frac{(4\pi)^\epsilon |p_z|^{-2\epsilon}}{16\pi^3\Gamma(1-\epsilon)} M_H^4(1-z_1)^2(1-z_2) \int_0^1 udu \int_0^1 dv\, d\phi_{n+1}(p_1,p_2;q,k_1,\ldots,k_m) d\phi_m(k';k_1',\ldots,k_m'). \tag{4.91}$$

Now we can repeat the argument of the previous section about the 2 phase spaces dependences on the dimension full variables. Therefore:

- $d\phi_{n+1}$ has a very similar form to the previous case, and its dependence is found to be

$$\frac{(\hat{s}-q^2)^2}{q^2} \sim 4(1-u)^2 M_H^2(1-z_1)^2. \tag{4.92}$$

- $d\phi_m$ has a very similar dependence, too, but it contains a different scale because what was a numeric factor $1 - z_2$ is now singular, hence

$$(k')^2 \sim uvM_H^2(1 - z_1)(1 - z_2). \tag{4.93}$$

We can read from the above relations what the soft/collinear scales are identical to the singly soft case: a soft scale $\Lambda_{DY}^2 = M_H^2(1 - z_1)^2$, and a collinear scale $\Lambda_{DIS}^2 = M_H^2(1 - z_1)(1 - z_2)$, but in section 4.6 and 4.7 we will highlight the differences both in their interpretations and in their behaviours. For, now, it is important to understand that, even if we allowed the possibility for both soft (all the components of the four-momentum are small) and collinear radiation (the transverse components of the four-momentum are small) to be emitted, actually the doubly soft limit case implies that also the longitudinal momentum is becoming small, so also the longitudinal component of the collinear radiation is becoming small, hence the extra radiation is globally becoming soft. Therefore, Λ_{DIS}^2, which was derived as a collinear scale, is actually a soft scale.

4.6 Doubly soft limit: the resummation formulas

In this section we apply the multi-scale resummation formulas to the hard and soft/collinear scales obtained in the doubly soft limit. Firstly, we show that the soft scale exponent produces only subleading contributions, therefore it can be excluded from the resummation formula. Secondly, we prove that the collinear scale $M_H^2(1 - z_1)(1 - z_2)$ is mapped in Mellin space in the scale $\frac{M_H^2}{N_1 N_2}$. Finally, we prove that the resummation formula with the remaining scale can be mapped into the resummation formula already known in literature. We also prove that in the doubly soft limit, the resummation formula for the rapidity distribution can be obtained from the one of the inclusive cross section by substituting $N \to N_1 N_2$.

We aim to apply the relations (3.34) and (3.35) to the case of rapidity distributions in the doubly soft limit, where it is required the knowledge of the hard scales and the soft scales. In the doubly soft limit, the only hard scale available is M_H^2, which is the endpoint of the kinematic. In section 4.5.3 we obtained the soft scale $M_H^2(1 - z_1)^2$ (M_H^2/N_1^2 in Mellin space) and the collinear scale $M_H^2(1 - z_1)(1 - z_2)$ ($M_H^2/(N_1 N_2)$ in Mellin space, proved in the section 4.6.2), where both z_1 and z_2 are approaching 1. Therefore, in this case, the resummation formula (3.34) becomes:

$$C\left(N_1, N_2, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2}, \alpha_s(M_H^2)\right)$$
$$\exp\left\{\int_1^{N_1^2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} \hat{g}_1(\alpha_s(k^2/n), N_2) + \int_1^{N_1 N_2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} \hat{g}_2(\alpha_s(k^2/n))\right\}. \tag{4.94}$$

In section 4.6.3 we will show that the above resummation formula is equivalent to:

$$C\left(N_1, N_2, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C^c\left(\frac{M_H^2}{\mu^2}, \alpha_s(M_H^2)\right)$$
$$\exp\left\{\int_0^1 \frac{z_1^{N_1-1}-1}{1-z_1}\int_{\mu^2}^{M_H^2(1-z_1)^2}\frac{d\lambda^2}{\lambda^2}g_1(\alpha_s(\lambda^2))\right.$$
$$+\int_0^1 dz_1\int_0^1 dz_2\,\frac{z_1^{N_1-1}z_2^{N_2-1}-1}{(1-z_1)(1-z_2)}g_2(\alpha_s(M_H^2(1-z_1)(1-z_2)))$$
$$\left.\Theta(M_H^2(1-z_1)-\mu^2)\Theta(M_H^2(1-z_2)-\mu^2)\right\}. \quad (4.95)$$

4.6.1 The suppressed scale

We now show that the scale $\Lambda_{DY} = M_H^2(1-z_1)^2$ gives origin only to subleading terms in Mellin-Mellin space. In order to do this, we calculate explicitly the Mellin-Mellin transform of a general contribution originated from this scale, which reads as follows:

$$\int_0^1 dz_2\, z_2^{N_2-1}\int_0^1 dz_1\, z_1^{N_1}\left[\frac{\ln^p(1-z_1)}{1-z_1}\right]_+$$
$$= \frac{1}{N_2}\int_0^1 dz_1\,\left(z_1^{N_1-1}-1\right)\left[\frac{\ln^p(1-z_1)}{1-z_1}\right] = O\left(\frac{\ln^p N_1}{N_2}\right). \quad (4.96)$$

Therefore, we have just shown that every contribution coming from the soft scale is suppressed as $N_2 \to \infty$. Therefore, in the formulas (4.117) and (4.118) the first terms in the exponential produce subleading contributions and we are left with:

$$C\left(N_1, N_2, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C_0\left(\frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right)$$
$$\exp\left\{\int_1^{N_1N_2}\frac{dn}{n}\int_{n\mu^2}^{M_H^2}\frac{dk^2}{k^2}\hat{g}_2(\alpha_s(k^2/n))\right\}, \quad (4.97)$$

and

$$C\left(N_1, N_2, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C_0\left(\frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right)$$
$$\exp\left\{\int_0^1 dz_1\int_0^1 dz_2\,\frac{z_1^{N_1-1}z_2^{N_2-1}-1}{(1-z_1)(1-z_2)}g_2(\alpha_s(M_H^2(1-z_1)(1-z_2)))\right.$$
$$\left.\Theta(M_H^2(1-z_1)-\mu^2)\Theta(M_H^2(1-z_2)-\mu^2)\right\}. \quad (4.98)$$

4.6.2 The remaining soft scale

We show that in Mellin-Mellin space the contributions coming from the scale $M_H^2(1-z_1)(1-z_2)$ are mapped into logarithms of the form $\ln^p(N_1N_2)$ and this

task will be accomplished by providing a generating functional. First, we take the double transform of the general contribution coming from the scale Λ^2_{DIS}:

$$I_p = \iint_0^1 dz_1\, dz_2\, z_1^{N_1-1} z_2^{N_2-1} \left[\frac{\ln^p((1-z_1)(1-z_2))}{(1-z_1)(1-z_2)}\right]_+ = \\ = \iint_0^1 dz_1\, dz_2\, \left(z_1^{N_1-1} z_2^{N_2-1} - 1\right) \frac{\ln^p((1-z_1)(1-z_2))}{(1-z_1)(1-z_2)}. \quad (4.99)$$

Then, as already stated, we introduce a generating functional $G(N_1, N_2, \eta)$, defined as

$$G(N_1, N_2, \eta) := \iint_0^1 dz_1\, dz_2\, \left(z_1^{N_1-1} z_2^{N_2-1} - 1\right) ((1-z_1)(1-z_2))^{\eta-1}, \quad (4.100)$$

from which I_p can be derived as follows:

$$I_p = \left[\frac{d^p}{d\eta^p} G(N_1, N_2, \eta)\right]_{\eta=0}. \quad (4.101)$$

We now recognise that the definition (4.100) can be written as

$$G(N_1, N_2, \eta) = \left[\int_0^1 dz_1\, z_1^{N_1-1}(1-z_1)^{\eta-1}\right] \left[\int_0^1 dz_2\, z_2^{N_2-1}(1-z_2)^{\eta-1}\right] \\ - \left[\int_0^1 dz_1\, (1-z_1)^{\eta-1}\right] \left[\int_0^1 dz_2 (1-z_2)^{\eta-1}\right] = \beta(N_1, \eta)\beta(N_2, \eta) - \frac{1}{\eta^2}, \quad (4.102)$$

where we used the definition of Euler β function in the first term, and we calculated the integrals in the second term. Our case is based on the assumption the $|N_1|, |N_2| \to \infty$, therefore we can apply the following Stirling approximation

$$\beta(N, \eta) = \frac{\Gamma(N)\Gamma(\eta)}{\Gamma(N+\eta)} \sim \frac{\Gamma(\eta)}{N^\eta} = \frac{1}{\eta}\frac{\Gamma(1+\eta)}{N^\eta} \quad (4.103)$$

to (4.102), obtaining:

$$= \frac{1}{\eta^2}\frac{\Gamma(1+\eta)^2}{(N_1 N_2)^\eta} - \frac{1}{\eta^2} = \frac{1}{\eta^2}\left(\frac{\Gamma(1+\eta)^2}{(N_1 N_2)^\eta} - 1\right). \quad (4.104)$$

This result proves that the dependence of the partonic rapidity distribution over dimension full variables can appear only through the combination of variables $N_1 N_2$ in Mellin-Mellin space. In fact, since the general contribution I_p is obtained as the derivative with respect to η of G, which is a function only of $N_1 N_2$, I_p itself can be only a function of $N_1 N_2$.

4.6.3 Equivalence between resummation formulas

Now we prove that formulas (4.97) and (4.98) are equivalent. The latter has already been proved in [16] using a different method. We start by considering a generic

power series in the running coupling constant as in the exponential of the formula (4.98):

$$g_2(\alpha_s(M_H^2(1-z_1)(1-z_2))) = \sum_{i=1}^{\infty} g_{2i}\alpha_s^i(M_H^2(1-z_1)(1-z_2))$$
$$= \sum_{p=0}^{\infty} \tilde{g}_p(\alpha_s(M_H^2)) \ln^p((1-z_1)(1-z_2)), \quad (4.105)$$

where g_{2i} are numeric coefficients, $\tilde{g}_p$ are functions only of $\alpha_s(M_H^2)$ and we have dropped the Heaviside Θ function because they only modify the domain of integration. Thanks to the linearity of the Mellin-Mellin transform, we can focus on the transform of a single logarithmic contribution, which can be obtained as in . Then, by looking at (4.101), we realise that we need to do some manipulations on the generating functional. We expand the formula provided by (4.104), in powers of η, as follows

$$G(N_1, N_1, \eta) = \frac{1}{\eta^2}\left(\frac{1}{(N_1N_2)^\eta}\sum_{k=0}^{\infty}\frac{(\Gamma^2)^{(k)}(1)}{k!}\eta^k - 1\right)$$
$$= \frac{1}{\eta^2(N_1N_2)^\eta} - \frac{1}{\eta^2} + \sum_{k=1}^{\infty}\frac{(\Gamma^2)^{(k)}(1)}{k!}\frac{\eta^k}{(N_1N_2)^\eta\eta^2}$$
$$= \frac{1}{\eta^2(N_1N_2)^\eta} - \frac{1}{\eta^2} + \sum_{k=1}^{\infty}\frac{(\Gamma^2)^{(k)}(1)}{k!}\frac{d^k}{d\ln^k\left(\frac{1}{N_1N_2}\right)}\left(\frac{1}{\eta^2(N_1N_2)^\eta} - \frac{1}{\eta^2}\right)$$
$$= \sum_{k=0}^{\infty}\frac{(\Gamma^2)^{(k)}(1)}{k!}\frac{d^k}{d\ln^k\left(\frac{1}{N_1N_2}\right)}\left(\frac{1}{\eta^2(N_1N_2)^\eta} - \frac{1}{\eta^2}\right). \quad (4.106)$$

In analogy with [13], we can give an integral representation of the function

$$\frac{1}{\eta^2(N_1N_2)^\eta} - \frac{1}{\eta^2} = \left[\int_{1-\frac{1}{N_1}}^{1} dz_1\,(1-z_1)^{\eta-1}\right]\left[\int_{1-\frac{1}{N_2}}^{1} dz_2\,(1-z_2)^{\eta-1}\right]$$
$$- \left[\int_0^1 dz_1\,(1-z_1)^{\eta-1}\right]\left[\int_0^1 dz_2\,(1-z_2)^{\eta-1}\right], \quad (4.107)$$

but this is not the most convenient one, because the 2 scales N_1 and N_2 are separated. It is more convenient to define 2 new variables x and w, where x is playing the equivalent of the soft variable x as in [13]. It is important to note that this change of variables is providing us with the soft scale and it could not have been done before the approximation (4.103) for $|N_1|, |N_2| \to \infty$, because only in this limit the 2 scales z_1 and z_2 are unifying into a single soft scale x. Having said that, in the first term we define $(1-z_1) = (1-w)/N_1$ and $(1-z_2) = N_1(1-x)$, whilst

in the second term we simply choose $w = z_1$ and $x = z_2$. The result is

$$= \left[\int_0^1 dw\,(1-w)^{\eta-1}\right]\left[\int_{1-\frac{1}{N_1N_2}}^1 dx\,(1-x)^{\eta-1}\right]$$
$$- \left[\int_0^1 dw\,(1-w)^{\eta-1}\right]\left[\int_0^1 dx\,(1-x)^{\eta-1}\right]$$
$$= -\left[\int_0^1 dw\,(1-w)^{\eta-1}\right]\left[\int_0^{1-\frac{1}{N_1N_2}} dx\,(1-x)^{\eta-1}\right], \quad (4.108)$$

we substitute it into (4.106) and we obtain

$$G(N_1,N_2,\eta) = -\sum_{k=0}^{\infty}\frac{(\Gamma^2)^{(k)}(1)}{k!}\frac{d^k}{d\ln^k\left(\frac{1}{N_1N_2}\right)}$$
$$\left[\int_0^1 dw\int_0^{1-\frac{1}{N_1N_2}} dx\,((1-w)(1-x))^{\eta-1}\right]. \quad (4.109)$$

Therefore, we substitute this new relation into (4.101) and we get:

$$I_p = -\sum_{k=0}^{\infty}\frac{(\Gamma^2)^{(k)}(1)}{k!}\frac{d^k}{d\ln^k\left(\frac{1}{N_1N_2}\right)}$$
$$\left[\int_0^1 dw\int_0^{1-\frac{1}{N_1N_2}} dx\,\frac{\ln^p((1-w)(1-x))}{(1-w)(1-x)}\right] + O\left(\frac{1}{N_i}\right). \quad (4.110)$$

Our aim is to change the derivates with respect to Mellin variables into the derivatives with respect to (x,w) space. We want to exploit relation 3.8 of [13]:

$$\frac{d^k}{d\ln^k(\frac{1}{N})}\int_0^{1-\frac{1}{N}} dx\,\frac{\ln^p(1-x)}{1-x} = \int_0^{1-\frac{1}{N}}\frac{dx}{1-x}\frac{d^k\ln^p(1-x)}{d\ln^k(1-x)}, \quad (4.111)$$

which is valid for the single variable case. Now we exploit the fact that

$$\ln^p((1-w)(1-x)) = (\ln(1-w)+\ln(1-x))^p = \sum_{q=0}^{p}\binom{p}{q}\ln^{p-q}(1-w)\ln^q(1-x) \quad (4.112)$$

during the following manipulation of (4.110) in order to apply (4.111):

$$
\begin{aligned}
I_p = &-\sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \sum_{q=0}^{p} \binom{p}{q} \int_0^1 dw \, \frac{\ln^{p-q}(1-w)}{1-w} \\
&\qquad \frac{d^k}{d\ln^k\left(\frac{1}{N_1 N_2}\right)} \int_0^{1-\frac{1}{N_1 N_2}} dx \, \frac{\ln^q(1-x)}{1-x} \\
= &-\sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \sum_{q=0}^{p} \binom{p}{q} \int_0^1 dw \, \frac{\ln^{p-q}(1-w)}{1-w} \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \frac{d^k \ln^q(1-x)}{d\ln^k(1-x)} \\
= &-\sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \int_0^1 \frac{dw}{1-w} \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \frac{d^k \ln^p((1-w)(1-x))}{d\ln^k(1-x)} \\
= &-\sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \int_0^1 \frac{dw}{1-w} \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \frac{d^k \ln^p((1-w)(1-x))}{d\ln^k((1-w)(1-x))}, \qquad (4.113)
\end{aligned}
$$

where in the last step we realised that it is possible to substitute the k-th derivative in the following way:

$$
\frac{d}{d\ln(1-x)} = \frac{d\ln((1-w)(1-x))}{d\ln(1-x)} \frac{d}{d\ln((1-w)(1-x))} = \frac{d}{d\ln((1-w)(1-x))}. \qquad (4.114)
$$

Now, thanks to (4.113), we are able to achieve the set goal. We start again from the Mellin-Mellin transform of (4.105) and we get:

$$
\begin{aligned}
\iint_0^1 dz_1 \, dz_2 \, \frac{z_1^{N_1-1} z_2^{N_2-1} - 1}{(1-z_1)(1-z_2)} g_2(\alpha_s(M_H^2(1-z_1)(1-z_2))) &= \sum_{p=0}^{\infty} \tilde{g}_p(\alpha_s(M_H^2)) I_p \\
&= -\sum_{p=0}^{\infty} \tilde{g}_p(\alpha_s(M_H^2)) \sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \int_0^1 \frac{dw}{1-w} \\
&\qquad \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \frac{d^k \ln^p((1-w)(1-x))}{d\ln^k((1-w)(1-x))} \\
&= \int_0^1 \frac{dw}{1-w} \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \frac{d^k}{d\ln^k((1-w)(1-x))} \\
&\qquad \left[-\sum_{p=0}^{\infty} \tilde{g}_p(\alpha_s(M_H^2)) \sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \ln^p((1-w)(1-x)) \right] \\
&= \int_0^1 \frac{dw}{1-w} \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \left[-\sum_{k=0}^{\infty} \frac{(\Gamma^2)^{(k)}(1)}{k!} \frac{d^k g_2(\alpha_s(M_H^2(1-w)(1-x)))}{d\ln^k((1-w)(1-x))} \right] \\
&= \int_0^1 \frac{dw}{1-w} \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \hat{g}_2(M_H^2(1-w)(1-x)), \qquad (4.115)
\end{aligned}
$$

where in the last step we defined a new power series $\hat{A}(\alpha_s(M_H^2(1-w)(1-x)))$ as the squared brackets in the step before. Now the remaining steps are straightforward:

we only need to change variables in the integral $k^2 = M_H^2(1-w)$ and $n = \frac{1}{1-x}$ and (4.115) results in:

$$= \int_1^{N_1 N_2} \frac{dn}{n} \int_0^{M_H^2} \frac{dk^2}{k^2} \hat{g}_2(\alpha_s(k^2/n)) \tag{4.116}$$

Therefore, we can simply use the resummation formulas for the inclusive cross section substituting $N \to N_1 N_2$.

4.7 Singly soft limit: the resummation formula

As in section 4.6, we apply the relations (3.34) and (3.35) to the case of rapidity distributions in the singly soft limit. In this case, there are 2 hard scales: M_H^2 and $M_H^2(1-z_2)$ (for z_2 fixed and $z_1 \to 1$), or M_H^2/N_2 in Mellin space. In section 4.5.3 we obtained the soft scale $M_H^2(1-z_1)^2$ (M_H^2/N_1^2 in Mellin space) and the collinear scale $M_H^2(1-z_1)(1-z_2)$ ($M_H^2/(N_1 N_2)$ in Mellin space, proved in the section 4.6.2), where both z_1 and z_2 are approaching 1. Therefore, in this case, the resummation formula (3.34) becomes:

$$C\left(N_1, \frac{M_H^2}{\mu^2}, \frac{M_H^2/N_2}{\mu^2}, \alpha_s(\mu^2)\right) = C_0\left(\frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right)$$
$$\exp\left\{\int_1^{N_1^2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2} \frac{dk^2}{k^2} g_1(\alpha_s(k^2/n), N_2) + \int_1^{N_1 N_2} \frac{dn}{n} \int_{n\mu^2}^{M_H^2/N_2} \frac{dk^2}{k^2} g_2(\alpha_s(k^2/n))\right\}, \tag{4.117}$$

and formula (3.35):

$$C\left(N_1, N_2, \frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right) = C_0\left(\frac{M_H^2}{\mu^2}, \alpha_s(\mu^2)\right)$$
$$\exp\left\{\int_0^1 dz_1 \frac{z_1^{N_1-1}-1}{1-z_1} \int_{\mu^2}^{M_H^2(1-z_1)^2} \frac{d\lambda^2}{\lambda^2} \hat{g}_1(\alpha_s(\lambda^2), N_2)\right.$$
$$\left. + \int_0^1 dz_1 \frac{z_1^{N_1-1}-1}{1-z_1} \int_{\mu^2}^{\frac{M_H^2}{N_2}(1-z_1)} \frac{d\lambda^2}{\lambda^2} \hat{g}_1(\alpha_s(\lambda^2))\right\}. \tag{4.118}$$

Since N_2 is no longer approaching ∞, the argument of section 4.6.1 can no longer be applied, therefore there are no suppressed scales.

Conclusion

In thesis thesis we considered the rapidity distribution of a colourless massive particle final state. In particular, we focused on the resummation of soft logarithms in two threshold regions: the doubly soft limit, i.e. the limit where the centre of mass energy of the collision is approaching its minimum possible value in order to produce the final state massive particle ($z \to 1$, or $z_1 \to 1$ and $z_2 \to 1$) and the singly soft limit, i.e. the limit where the partonic longitudinal rapidity of the final state massive particle is approaching its maximum possible value (z_2 fixed and $z_1 \to 1$ or z_1 fixed and $z_2 \to 1$).

In order to derive the resummation formulas, we applied the general formulas for multi-scale resummation, which require the knowledge of the hard scales and the soft/collinear scales of the process.

We argued that in the doubly soft limit, the process has only one hard scale, i.e. the mass of the final state particle M_H^2. On the other hand, in the singly soft limit the process has two hard scales: the mass of the final state particle M_H^2, and the longitudinal momentum endpoint, which is $M_H^2(1-z_2)$ for z_2 fixed and $z_1 \to 1$.

Then, thanks to an analysis of the structure of the phase space, we derived the soft scale $M_H^2(1-z_1)^2$ and the collinear scale $M_H^2(1-z_1)(1-z_2)$. They are the same both in the doubly and singly soft limits, but with different behaviours and interpretations.

Moreover, we proved that in the doubly soft limit, the soft scales produces only subleading contributions, therefore we are left with the collinear scale, which is actually playing the role of a soft scale because in the doubly soft limit the collinear radiation is also becoming soft.

We proved that the doubly soft resummation formula can be rewritten in a different form, which has already been obtained with a different method. We also proved that the structure of the resummation formula is identical to the one of the inclusive cross section, and can be obtained from it by substituting $N \mapsto N_1 N_2$.

To summarize, the single-scale renormalization group approach, together with the phase space structure analysis, in [13] has been used to derive the resummation formula for the inclusive cross section. The multi-scale approach proved to be effective in deriving both the transverse momentum resummation formulas [15] and the rapidity distributions resummation formulas (object of this thesis). The natural following application is the fully differential cross section, both in transverse momentum and in rapidity, namely the one where the Higgs kinematic is completely fixed.

Appendix A

Mathematical tools

A.1 Mellin transform

Here we provide the definition and some properties of the mathematical objects used in this thesis. First of all, we define the *monolateral Laplace transform*. Given a function $f(t)$ with $t \in [0, \infty]$, its monolateral Laplace transform is defined as follows:

$$\tilde{f}(N) = \mathcal{L}[f](N) := \int_0^\infty dt\, e^{-tN} f(t). \tag{A.1}$$

Given the Laplace transform $\tilde{f}(N)$, $f(t)$ can be obtained as its *inverse* Laplace transform:

$$f(t) = \mathcal{L}^{-1}[\tilde{f}](t) = \int_{c-i\infty}^{c+i\infty} dN\, e^{tN} \tilde{f}(N), \tag{A.2}$$

where c is a real number chosen to be greater to the real parts of every pole of the function $\tilde{f}(N)$. Sometimes, it can be useful also to define the *bilateral* Laplace transform:

$$\mathcal{L}_B[f](N) := \int_{-\infty}^{\infty} dt\, e^{-tN} f(t). \tag{A.3}$$

In definition (A.1), by changing the integrated variable $t \mapsto x = -\ln t$, we obtain the definition of *Mellin transform*, which is defined for function $f(x)$ with $x \in [0, 1]$:

$$\tilde{f}(N) = \mathcal{M}[f](N) := \int_0^1 dx\, x^{N-1} f(x), \tag{A.4}$$

and analogously its inverse is:

$$f(x) = \mathcal{M}^{-1}[\tilde{f}](x) = \int_{c-i\infty}^{c+i\infty} dN\, x^{-N} \tilde{f}(N). \tag{A.5}$$

Like Fourier and Laplace transforms, it is possible to define a *convolution* which factorize under Mellin transform. The convolution of 2 functions $f(x)$ and $g(x)$ is defined as follows:

$$(f \otimes g)(x) := \int_x^1 \frac{dy}{y} f(y) g\left(\frac{x}{y}\right). \tag{A.6}$$

Through a simple manipulation, it is possible to rewrite the above definition in the following form:

$$(f \otimes g)(x) \coloneqq \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z). \tag{A.7}$$

Expression (A.7) allows us to simply generalise the convolution of 2 functions to the case of many functions, which reads as follows:

$$(f_1 \otimes ... \otimes f_n)(x) \coloneqq \int_0^1 dx_1 ... \int_0^1 dx_n\, f_1(x_1)...f_n(x_n)\delta(x - x_1...x_n). \tag{A.8}$$

The factorization property of the convolution is very easy to prove, in fact:

$$\mathcal{M}[(f \otimes g)](N) = \int_0^1 dx\, x^{N-1} \left[\int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z)\right]$$
$$\left[\int_0^1 dy\, y^{N-1} f(y)\right] \left[\int_0^1 dz\, z^{N-1} g(z)\right] = \tilde{f}(N)\tilde{g}(N), \tag{A.9}$$

and, in complete analogy, in the many functions case we get

$$\mathcal{M}[(f_1 \otimes ... \otimes f_n)](N) = \tilde{f}_1(N)...\tilde{f}_n(N). \tag{A.10}$$

A.2 Plus distributions

Another important tool is the *plus distribution*, which is a distribution in the sense that it is a map from a function space to the real numbers. Therefore it is defined by its action on a test function of this function space:

$$\int_0^1 dx\, [f(x)]_+ g(x) \coloneqq \int_0^1 dx\, f(x)(g(x) - g(1)). \tag{A.11}$$

It immediately follows from the definition that every constant function in $[0, 1]$ is mapped onto 0. Particularly useful is the following identity:

$$\int_0^1 dx\, [f(x)]_+ = 0. \tag{A.12}$$

An equivalent definition of the plus distribution can be provided by the limit of a class of distributions as follows:

$$[f(x)]_+ \coloneqq \lim_{\epsilon \to 0^+} \left[\theta(1 - x - \epsilon) - \delta(1 - z) \int_0^{1-\epsilon} dx\, f(x)\right]. \tag{A.13}$$

Appendix B

A proof that $u = (1 + \cos\theta)/2$

The partonic cross section differential in rapidity, which has been written above as function of the variables z and $\hat{y}$, in literature is often presented in terms of the variables z and

$$u = \frac{z_2(1 - z_1^2)}{(1 - z_1 z_2)(z_1 + z_2)} \tag{B.1}$$

and their plus distributions. While the meaning of the former has been already explained, one might wonder why is that the latter is such a convenient variable. The answer is that u is nothing but the angle at which a radiation particle is emitted in the process we are considering.

Here we give a brief proof of this relation. To be as clear as possible, we report the four momenta of the considered particles

$$p_1 + p_2 = \sqrt{\hat{s}}(1, 0, 0, 0)$$

$$p_H = \left(\sqrt{M_H^2 + p_T^2}\cosh\hat{y}, \vec{p}_T, \sqrt{M_H^2 + p_T^2}\sinh\hat{y}\right)$$

$$X = \left(\sqrt{M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y}}, -\vec{p}_T, -\sqrt{M_H^2 + p_T^2}\sinh\hat{y}\right),$$

and we want to recognise in the expression

$$u = \frac{z_2(1 - z_1^2)}{(1 - z_1 z_2)(z_1 + z_2)} \tag{B.2}$$

elements of the four-momenta above. Firstly, we substitute $z_1 = \sqrt{z}e^{\hat{y}}$ and $z_2 = \sqrt{z}e^{-\hat{y}}$ and we obtain:

$$u = \frac{e^{-\hat{y}}(1 - ze^{2\hat{y}})}{2(1 - z)\cosh\hat{y}} = \frac{e^{-\hat{y}} - ze^{\hat{y}}}{2(1 - z)\cosh\hat{y}} \tag{B.3}$$

Then, we express the exponential as $e^{\hat{y}} = \cosh\hat{y} + \sinh\hat{y}$ and $e^{\hat{y}} = \cosh\hat{y} - \sinh\hat{y}$ to make the expression more similar to the four-momenta, obtaining:

$$u = \frac{\cosh\hat{y} - \sinh\hat{y} - z(\cosh\hat{y} + \sinh\hat{y})}{2(1 - z)\cosh\hat{y}} = \frac{1}{2}\left[1 - \frac{(1 + z)\sinh\hat{y}}{(1 - z)\cosh\hat{y}}\right]. \tag{B.4}$$

Thirdly, we switch from the variable z to the variables $\hat{s}$ and M_H^2 and we get:

$$u = \frac{1}{2}\left[1 - \frac{\hat{s} + M_H^2}{\hat{s} - M_H^2}\frac{\sinh\hat{y}}{\cosh\hat{y}}\right]. \tag{B.5}$$

In (B.5) we can not immediately recognise elements of the four-momenta above, but the simple trick is to write the conservation of energy:

$$\sqrt{\hat{s}} = \sqrt{M_H^2 + p_T^2}\cosh\hat{y} + \sqrt{M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y}}. \tag{B.6}$$

Firstly, we square this equation firstly as

$$\left(\sqrt{\hat{s}} - \sqrt{M_H^2 + p_T^2}\cosh\hat{y}\right)^2 = M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y} \tag{B.7}$$

and then as

$$\left(\sqrt{\hat{s}} - \sqrt{M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y}}\right)^2 = (M_H^2 + p_T^2)\cosh^2\hat{y}. \tag{B.8}$$

Equations (B.7) and (B.8) can be rewritten as

$$\hat{s} + M_H^2 = 2\sqrt{\hat{s}}\sqrt{M_H^2 + p_T^2}\cosh\hat{y} \tag{B.9}$$

and

$$\hat{s} - M_H^2 = 2\sqrt{\hat{s}}\sqrt{M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y}}. \tag{B.10}$$

By taking the ratio of the 2 relations above, we obtain

$$\frac{\hat{s} + M_H^2}{\hat{s} - M_H^2} = \frac{\sqrt{M_H^2 + p_T^2}\cosh\hat{y}}{\sqrt{M_H^2\sinh^2\hat{y} + p_T^2\cosh^2\hat{y}}}, \tag{B.11}$$

which, once substituted in (B.5), allow us to see that u is nothing but

$$u = \frac{1}{2}\left(1 + \frac{X^z}{X^0}\right) = \frac{1 + \cos\theta}{2}. \tag{B.12}$$

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