

# UNIVERSITÀ DEGLI STUDI DI MILANO FACOLTÀ DI SCIENZE E TECNOLOGIE 

Corso di Laurea Magistrale in fisica

# Towards fully differential soft resummation 

Relatore: Stefano Forte
Correlatore: Marco Zaro

Tesi di Laurea di:
Cesare Carlo Mella
Matr. 960065

## Contents

1 QCD and the Parton Model ..... 9
1.1 QCD lagrangian ..... 9
1.1.1 Gauge theory quantization ..... 12
1.2 Running of the coupling: asymptotic freedom ..... 14
1.3 The parton model ..... 17
1.4 QCD corrections to DIS ..... 19
1.5 DGLAP evolution equations ..... 22
1.6 Hadron collisions ..... 23
2 Resummation theory ..... 27
2.1 Large Logarithms and Resummation ..... 27
2.1.1 Soft Large Logarithms: the origin ..... 27
2.1.2 Soft Large Logarithms: resummation ..... 28
2.2 General Resummation Theory ..... 30
2.3 Transverse Momentum Distribution ..... 32
2.4 Rapidity Distribution ..... 35
2.4.1 Rapidity distribution in Mellin-Mellin space ..... 39
3 The fully differential distribution ..... 45
3.1 Higgs production in gluon fusion: the effective interaction ..... 45
3.2 Kinematics and Notations ..... 46
3.2.1 Notations ..... 46
3.2.2 Kinematic ..... 48
3.3 Fully differential distribution at NLO and its Mellin-Fourier transform ..... 50
3.3.1 The LO ..... 51
3.3.2 $\mathcal{O}\left(\alpha_{s}\right)$ corrections ..... 52
3.4 Fully differential distribution at NLO: threshold behaviour ..... 54
3.4.1 Threshold evaluation of regular contribution ..... 55
3.4.2 Change of variable of singular contribution ..... 55
4 Fully differential threshold behaviour in Mellin-Fourier space ..... 59
4.1 The small $p_{T}$ limit ..... 60
4.1.1 Threshold evaluation of small- $p_{T}$ distribution ..... 61
4.2 Threshold behaviour for fixed $p_{T}$ ..... 65
4.3 A conjecture for the resummed cross section ..... 67
5 Conclusion ..... 71
A Plus Distributions ..... 73
A. 1 Mellin Transform ..... 74
A.1.1 Convolution and Mellin transform ..... 74
A. 2 The Plus distribution ..... 75
A.2.1 A useful identity ..... 76
A. 3 Large-N behaviour of the Mellin transform ..... 77
A.3.1 The Mellin transform of Plus distribution ..... 78
B Changes of variables ..... 81
B. 1 Changes of variables in plus distributions ..... 82
B.1.1 Change of variable ..... 82
B.1.2 Some other identities ..... 84
B. 2 Other Integrals ..... 85
B. 3 The $p_{T} \rightarrow 0$ limit ..... 86
B. 4 Integrals for the Final Expansions ..... 87
C Special Functions ..... 91
C. 1 The Euler Gamma function and its derivatives ..... 92
C. 2 Gamma Related Functions ..... 94

## Introduction

In the last fifteen years particle physicists have been provided with an enormous amount of data. At CERN, the Large Hadron Collider (LHC) started in 2008 and subsequent upgrades led the machine to increasingly high center of mass energies, reaching 7 TeV in 2010 and, then, the peak of 13 TeV in 2015. Data from LHC have permitted for deep and meticulous tests of the Standard Model, culminating with the observation of the Higgs Boson. It was the last missing piece for a consistent formulation of the theory. Even if deviations from Standard Model predictions have never been measured so far, we know that it does not answer to all our questions. Gravity is not included and it also fails in explaining the hierarchy of interaction strengths and particle masses. Moreover, Cosmological observations suggest the presence of Dark Matter and Dark Energy, that have so far evaded our subatomic experiments.

Through particle physicists, it is a shared opinion that we miss of a proper phenomenological input. Consequently, we are particularly interested in producing high precision theoretical calculations and experimental measurements within the Standard Model. Cross sections are the quantities we are mainly interested in. A cross section tells us the expected number of events of a certain kind in a particle collision experiment. Firstly, efforts had been mainly directed in total cross sections, which are usually referred to as completely inclusive cross sections. From a theoretical point of view, such observables are the easiest we can deal with. Moreover, these are the situations in which experiments provide the maximum possible statistics. Subsequently, the focus has shifted on more exclusive observables, namely differential cross sections. A differential distribution carries more theoretical information than its inclusive counterpart, but it is more difficult to compute. Experimentally, the price is a poorer statistics. The remedy for the latter will be provided by other upgrades of LHC, precisely aiming at increasing the number of events observed. After the third run of LHC, whose
start is scheduled for the next months, another long shutdown will be used for the installation of the High-Luminosity upgrade, which will increase the total number of events (Luminosity) by almost a factor of ten.

In this thesis, we are interested in theoretical predictions. We consider high energy processes in QCD with colourless final states, such as Higgs production or the Drell-Yan. The case of strongly interacting final states is more complicated and we will not discuss it. In the Drell-Yan process, a lepton pair is produced with a gauge boson as intermediate state. In this thesis we focus on the case of Higgs production, whose cross section is computed in the framework of an effective field theory, in which the top quark mass is considered much bigger than any other scale.

Cross sections in Quantum Field Theory (QFT) are calculated by means of perturbation theory. The final result is written as a power expansion with respect to the coupling constant, truncated at some perturbative order. The result of such a calculation is usually called a fixed order cross section. If the expansion parameter is small, then higher orders should give corrections of steadily decreasing importance. And indeed one way of reducing the theoretical uncertainty is compute a new perturbative order. Sometimes this is not sufficient. In some kinematical regions, the perturbative series is spoiled by the presence of so called large logarithms. Whenever a process is characterized by two or more different scales, these usually appears through their ratio in logarithmic contributions. If two, or more, of these scales become very different from each other, logarithms grows in an uncontrolled way, spoiling the convergence of the perturbative series. These large logarithms need to be resummed to all perturbative order.

Resummation Theory is the solution to this problem. We are particularly interested in Threshold Resummation, devoted to the summation of logarithms arising in the threshold region. The threshold kinematic configuration is defined by a partonic centre of mass energy which is just enough to produce the final state. An emitted gluon is obliged having vanishing energy, that is it becomes soft. The energy of a soft gluon and the invariant mass of the final state are very different and it gives rise to large logarithms. Threshold resummation for inclusive observables has been achieved long ago. Results are also available for distribution differential in rapidity, $Y$, or transverse momentum, $p_{T}$, of the final state. On the contrary, there are no available resummation prescriptions for completely differential distributions,
that is differential in both, the transverse momentum and rapidity:

$$
\frac{d^{2} \sigma}{d p_{T} d y}
$$

The purpose of this thesis is understanding the general structure of soft large logarithms in completely differential distribution. We organize the work as follows. Chapter 1 consists of a general overview of QCD. We present the parton model in its historical formulation and discuss the effect of QCD corrections to the LO. In Chapter 2 we describe threshold resummation. We want to highlight the general strategy and present known results. In particular, we introduce the Mellin and the Mellin-Fourier transform as tools to factorize the partonic cross section from the PDFs and to single out the threshold region. These transforms need to be taken with respect to proper partonic variables. Then, we describe resummation formulae for inclusive and single differential cross sections. In Chapter 3 we consider the completely differential distribution for Higgs production at NLO, using results from ref.[16]. We describe the kinematic of the process and its factorization properties under a Mellin-Fourier transform. Understanding the kinematic is a fundamental step in order to find the proper soft limit. We expect soft large logarithms to appear, in Mellin-Fourier reciprocal space, as logarithms of the conjugated variables $(N, M)$. We then perform a change of variable of the NLO partonic cross section, expressing it as a function of the Mellin-Fourier integration variables. In Chapter 4 we consider the problem of explicitly computing the double transform. We presents three results. First, we compute a double Mellin transform of the rapidity cross section at NLO and verify that it agrees with the - already known - general resummation formula. We note that a double Mellin transform is simply related to the Mellin-Fourier one by a linear transformation of the reciprocal space variables. Then, we come back to the fully differential distribution and compute the double transform by first taking the small- $p_{T}$ limit. The small- $p_{T}$ simplify the kinematic, since it approaches the $p_{T}$ integrated one. Finally, we deal with the full problem of fixed $p_{T}$ and $Y$. We perform a suitable expansion of the integrand and then we compute the double transform on the first relevant order. We formulate a conjecture for the complete series and make a check of consistency with our first order result.

## Chapter 1

## QCD and the Parton Model

This chapter is devoted to a general review of basic Quantum Chromodynamics. Our intention is to set up the framework for the next chapters, trying to outline the general structure of a perturbative calculation. Consequently, we do not give proofs of the mentioned theorems and we do not make explicitly computations either. The interested reader is referred to literature.

### 1.1 QCD lagrangian

Quantum ChromoDynamics (QCD) is the quantum field theory that describes strong interactions in the contest of the Standard Model of elementary particles. QCD is formulated as a gauge theory, with $S U(3)$ gauge symmetry group. It introduces spin- $1 / 2$ particles, called quarks, as elementary constituents of hadronic matter. Interactions are mediated by gluons, gauge bosons described by vector fields. The lagrangian of the theory is given by

$$
\begin{equation*}
\mathcal{L}=\sum_{\text {flavour }} \bar{q}_{a}\left(i \not D-m_{q}\right)_{a b} q_{b}-\frac{1}{4} F_{\mu \nu}^{a} F^{\mu \nu, a}, \tag{1.1}
\end{equation*}
$$

where repeated indexes are supposed to be summed over. We have introduced some operators:

- $q_{a}$ is the Dirac spinor field associated to quark and its antiparticle, with colour charge index $a$.
- The sum over flavour has to be understood as a sum over different $q$ fields.
- $D_{\mu}=\partial_{\mu}-i g_{s} \lambda^{a} A_{\mu}^{a}$ is the covariant derivative, where $A_{\mu}^{a}, a=1 \ldots 8$, are the vector gauge fields representing gluons, $\left\{\lambda^{a}\right\}$ is a basis for a representation of $s u(3)$ Lie algebra and $g_{s}$ is the theory coupling constant. ${ }^{1}$
- The field strength tensor is given by $F_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g_{s} f^{a b c} A_{\mu}^{b} A_{\nu}^{c}$.

Any matrix representation of a Lie algebra must satisfy commutation relations

$$
\begin{equation*}
\left[t^{a}, t^{b}\right]=i f^{a b c} t^{c} \tag{1.2}
\end{equation*}
$$

The set of numbers $f^{a b c}$ are called the structure constants of the algebra ${ }^{2}$ and the $t^{a}$ 's are elements of the algebra.

Importantly, in the field strength tensor there is a term which is bilinear in fields and proportional to the structure constants. It leads to interaction vertices with three or four gluons. Such term vanishes in an abelian theory. As a consequence, whereas gluons may interact with each others, photons cannot, being electromagnetic interactions built with the $U(1)$ symmetry group, which is abelian.

We have not said yet what the attribute gauge means. First, we require that quark fields transform in the fundamental three-dimensional unitary representation of $\operatorname{SU}(3)$, for which the set of matrices $\left\{\lambda_{a}\right\}$ may be chosen as the well-known Gell-Mann matrices. More in general, a $n$-dimensional object is said to transform in the $n$-dimensional representation of a group $G$ if it obeys the rule

$$
\begin{equation*}
\psi_{i} \longmapsto \psi_{i}^{\prime}=[g]_{i j} \psi_{j} \quad \forall g \in G, \tag{1.3}
\end{equation*}
$$

where $g$ is the abstract group element and $[g]_{i j}$ is its matrix representation.
For each quark fields, that is for each flavour, we consider a local version of this transformation law

$$
\begin{equation*}
\psi_{i}(x) \longmapsto \psi_{i}^{\prime}(x)=\exp \left(i \theta^{a}(x) \lambda^{a}\right)_{i j} \psi_{j}(x) . \tag{1.4}
\end{equation*}
$$

Here $\theta^{a}(x)$ is a function which selects an element of the group for any spacetime point. Eq.(1.4) if what we call a (local) gauge transformation. Note

[^0]that the group elements have been written as the exponential of suitable algebra counterparts.

The lagrangian in Eq.(1.1) is built as a gauge invariant quantity, namely its variation vanishes under the local action of $S U(3)$, provided that also the $A^{a}{ }_{\mu}$ fields transform accordingly. In particular they must transform in the eight-dimensional - adjoint - representation.
This is how a gauge theory is built in general: given a unitary representation of a symmetry group and an array of Dirac fermions which transform locally as in Eq.(1.4), then the requirement of invariance of the lagrangian forces the introduction of corrective gauge fields transforming in the adjoint representation.

The adjoint representation has dimension which is equal to the group dimension. As a consequence, the number of gauge bosons is fixed by the dimensionality of the group. For strong interactions $\operatorname{dim}\left(G_{Q C D}\right)=$ $\operatorname{dim}(S U(3))=8$ and we have eight different gluons $g$. The other Standard Model example is given by Electroweak interactions, with $G_{E W}=$ $S U(2) \otimes U(1)$ symmetry group. Since $\operatorname{dim}\left(G_{E W}\right)=4$ then there are four gauge bosons: the photon $\gamma$, the $Z^{0}$ and the charge couple $W^{ \pm}$.
A basis for the adjoint representation is given by

$$
\begin{equation*}
\left[T_{a}\right]_{b c}=-i f_{a b c} . \tag{1.5}
\end{equation*}
$$

Its a general result in Lie Group theory that the structure constants give a full description of the adjoint representation. The last thing to do is giving a proper dynamic to gauge fields. This is what the field strength tensor term provide for.
It is not difficult, though not very insightful, to verify the invariance of Eq.(1.1). This is made particularly simple if one considers infinitesimal transformations.
The important message here is that this local invariance principle underpins the construction of each theories in the Standard Model.

The number of flavour is a phenomenological input: there are $n_{f}=6$ flavour of quarks, carrying fractional charges and different in masses. We will consider high energy processes in which all quarks' masses are negligible, apart from $m_{t o p}$ which is quite large if compared with the others. ${ }^{3}$

[^1]| flavour | d | u | s | c | b | t |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| name | down | up | strange | charme | bottom | top |
| charge | $-1 / 3$ | $2 / 3$ | $-1 / 3$ | $2 / 3$ | $-1 / 3$ | $2 / 3$ |
| mass | $\approx 5.0 \mathrm{Mev}$ | $\approx 2.5 \mathrm{Mev}$ | $\approx 0.1 \mathrm{Gev}$ | $\approx 1.3 G e v$ | $\approx 4.2 G e v$ | $\approx 173 \mathrm{Gev}$ |
|  |  |  |  |  |  |  |

Table 1.1: Quarks in the Standard Model

### 1.1.1 Gauge theory quantization

It is far from the purposes of this thesis to dwell in a detailed treatment of gauge theories quantization procedure, which is a rather sophisticated topic, especially if a non abelian symmetry is involved, as in our case. We just want to outline the difficulties, outline the general solution and explain the main consequences. For more details, we refer the Reader to Refs. [13, $22]$.

Although the gauge principle is a very powerful constructive tool, it leads to some trouble in the quantization procedure. This becomes completely manifest when the functional approach is pursued. The naive application of the Feynman path integral in calculating the gauge boson propagator does not work. In particular, it is not possible to invert the quadratic form which defines it. This is due to the fact that we are trying to perform a sum - functional integration - over each possible field configurations, also gauge equivalent ones, which we cannot distinguish, in the sense that the weight of the functional sum is constant over such configurations. In other words, we are just integrating a constant over an infinite domain, the "volume" of the gauge group. First, we make the following observation: if the gauge group is supposed to represent a symmetry, it should be impossible to detect a gauge-transformed fields configuration. Observable must be gauge invariant quantities. As a consequence, it is our right to just "fix the gauge" 4 , and we expect the result to be independent from our fixing choice.

There is a very simple and illuminating way of looking at this and it is represented by the discrete case. Consider the following two dimensional

[^2]integral
\[

$$
\begin{align*}
I(j) & =\frac{1}{I_{0}} \iint_{\mathbb{R}^{2}} d x d y e^{-\frac{1}{2} b x^{2}+j x}, \\
I_{0} & =\iint_{\mathbb{R}^{2}} d x d y e^{-\frac{1}{2} b x^{2}} \tag{1.6}
\end{align*}
$$
\]

$I(j)$ is clearly ill defined in both the normalization $I_{0}$ and the numerator. $I(j)$ could represent the generating function for a statistical theory, with weight $e^{-\frac{1}{2} b x^{2}}$. The statistical moments would be given by

$$
\begin{equation*}
<x^{n}>=\left.\frac{d^{n}}{d j^{n}} I(j)\right|_{j=0} \tag{1.7}
\end{equation*}
$$

A translation along the $y$ axis is irrelevant to the physics, in the sense that it does not change the statistical weight, exactly as gauge transformations do. We can just get rid of this nonphysical degree of freedom. There are different ways of proceeding, all giving the same result: we could limit the $y$-integration over a compact domain or we could add a convergence factor, such as $e^{-\xi y^{2}}$. The easiest one is probably the following

$$
\begin{align*}
I^{r e g}(j) & =\left(I_{0}^{r e g}\right)^{-1} \iint_{\mathbb{R}^{2}} d x d y \delta\left(y-y_{0}\right) e^{-\frac{1}{2} b x^{2}+j x} \\
I_{0}^{r e g} & =\iint_{\mathbb{R}^{2}} d x d y \delta\left(y-y_{0}\right) e^{-\frac{1}{2} b x^{2}} \tag{1.8}
\end{align*}
$$

which just eliminates the integral over $y$. Feynman path integral is just the continuous limit of a the $N$-dimensional generalization of this last example. We mentioned above, in the case of the functional integral, about an operator which we could not invert. We can also give the same interpretation to Eq.(1.6). It is known that the the $n$-dimensional purely Gaussian integral is given by

$$
\begin{equation*}
\int_{\mathbb{R}^{n}} d x^{n} e^{-\vec{x} A \vec{x}}=\frac{(2 \pi)^{n / 2}}{\sqrt{\operatorname{det}(A)}} \tag{1.9}
\end{equation*}
$$

where $A$ must a $n \times n$ invertible matrix, otherwise $\operatorname{det}(A)=0$. Now, the normalization in Eq.(1.6) can be recast in the form

$$
\begin{equation*}
I_{0}=\iint_{\mathbb{R}^{2}} d x d y e^{-\frac{1}{2} \vec{x} B \vec{x}} \tag{1.10}
\end{equation*}
$$

where

$$
B=\left(\begin{array}{ll}
b & 0  \tag{1.11}\\
0 & 0
\end{array}\right)
$$

which is clearly not invertible.
The case of a functional integral is far more complicated and we are not going to reproduce here the gauge fixing procedure. A careful computation leads, $[12,22]$, to the following adjustments in the QCD lagrangian

$$
\begin{align*}
& \mathcal{L}_{\text {fixing }}=-\frac{1}{2 \xi}\left(\partial_{\mu} A_{a}^{\mu}\right)^{2}  \tag{1.12}\\
& \mathcal{L}_{\text {ghost }}=-\bar{c}_{a}\left(\partial_{\mu} D_{a b}^{\mu}\right) c_{b}
\end{align*}
$$

We have introduced two novelties: $c_{a}$ fields are called Faddeev-Popov ghost and they are anticommuting scalar fields. They are not physical, meaning that we cannot observe ghost particles. Nonetheless, they play an important role in giving physical gauge invariant results ${ }^{5}$. This last assertion leads us to the other new parameter, $\xi$, which is completely arbitrary and parametrizes the gauge choice. The final, measurable, result must be independent from $\xi$. It is possible to prove, at any order in perturbation theory, that this is exactly the case, provided ghosts are taken into account.

### 1.2 Running of the coupling: asymptotic freedom

Observables in particle physics are computed by means of perturbation theory. The main idea can be summarized as follows: the power expansion of observable with respect to some small dimensionless parameter, may be truncated at some fixed order, still giving an accurate prediction. Higher orders represent corrections to the approximate result.
The coupling constant of the theory is usually chosen to be the expansion parameter. This is justified in QED, where $\alpha_{e m} \approx \frac{1}{137}$. On the contrary, at first sight QCD seems to preclude any perturbative treatment, at least at small energy, where the coupling $\alpha_{s}$ does not satisfy the above condition. In order to find the solution to this issue, we first need to investigate deeper perturbation theory.
Consider an observable, $\hat{O}$, computed as power series of the coupling. In general, $\hat{O}$ should depend on the coupling itself, $\alpha$, the kinematic of the process and properties of the particles involved, mainly their masses and spins. In general, we write

$$
\begin{equation*}
\hat{O}=\hat{O}\left(\alpha,\left\{p_{i}, m_{i}, s_{i}, \ldots\right\}\right), \tag{1.13}
\end{equation*}
$$

[^3]and the formal perturbative series in readily given
\[

$$
\begin{equation*}
\hat{O}=\sum_{n=1}^{\infty} \hat{O}_{n}\left(\left\{p_{i}, m_{i}, s_{i}, \ldots\right\}\right) \alpha^{n} . \tag{1.14}
\end{equation*}
$$

\]

For the sake of discussion, suppose we want to deal with a process in which all energy scales, apart from one that we call $Q$, are negligible. The general expression for the observable reduces to $\hat{O}=\hat{O}(Q ; \alpha)$. If we make the further hypothesis that $\hat{O}$ is a dimensionless quantity, then we should write

$$
\begin{equation*}
\hat{O}=\hat{O}(\alpha), \tag{1.15}
\end{equation*}
$$

that is $\hat{O}$ has reduced to a constant to be computed perturbatively.
It is well known that corrections to the tree level ${ }^{6}$ usually give rise to divergent integrals for loop diagrams, due to the UV behaviour of the involved expressions. The problem has been resolved long ago, first operationally, by Feynman and others, and then conceptually, by Kenneth Wilson, see [26]. The procedure through which we deal with these infinities is called renormalization. One of the main consequences of renormalization is the introduction of an auxiliary, although completely arbitrary, energy scale $\mu$. Equipped with a new scale, we can no longer rely on Eq.(1.15). It is possible to construct the dimensionless ratio $Q^{2} / \mu^{2}$ on which $\hat{O}$, as well as the coupling, may depend. Instead of Eq.(1.15), we shall write

$$
\begin{equation*}
\hat{O}=\hat{O}\left(\frac{Q^{2}}{\mu^{2}} ; \alpha\left(\mu^{2}\right)\right) \tag{1.16}
\end{equation*}
$$

The important message here is that in renormalized quantities a new scale dependence may appear. In this case, $\hat{O}$ depends on $Q^{2}$. This has been observed experimentally, for example in the ratio between the rate of production of hadrons and that of charged leptons, in electron-positron annihilation, [12].
Remarkably, the coupling constant acquire a scale dependence too. It is necessary to proceed carefully: a dimensionless observable may acquire a scale dependence, but the $\mu$ dependence should disappear in the final result ${ }^{7}$. In fact, being $\mu$ a completely arbitrary scale, nothing that is observable can depend on it.

[^4]We express this by requiring the following differential equation to hold beyond perturbation theory

$$
\begin{equation*}
\mu^{2} \frac{d}{d \mu^{2}} \hat{O}\left(\frac{Q^{2}}{\mu^{2}} ; \alpha\left(\mu^{2}\right)\right)=\left[\mu^{2} \frac{\partial}{\partial \mu^{2}}+\beta\left(\alpha\left(\mu^{2}\right)\right) \frac{\partial}{\partial \alpha\left(\mu^{2}\right)}\right] \hat{O}=0, \tag{1.17}
\end{equation*}
$$

where we have introduced the beta function

$$
\begin{equation*}
\beta\left(\alpha\left(\mu^{2}\right)\right):=\frac{d}{d \log \left(\mu^{2}\right)} \alpha\left(\mu^{2}\right) \tag{1.18}
\end{equation*}
$$

Eq.(1.17) is known as Callan-Symanzik equation and it plays an important role in Quantum Field Theory. The beta function tells us about the scale dependence of the coupling and can be computed perturbatively as a power series in the coupling itself

$$
\begin{equation*}
\beta\left(\alpha\left(\mu^{2}\right)\right)=-\beta_{0} \alpha^{2}-\beta_{1} \alpha^{3}+\ldots \tag{1.19}
\end{equation*}
$$

It turns out that $\beta$ is positive for QED, but it is negative in QCD . The electromagnetic coupling is an increasing function of the energy, whereas the strong coupling reduces as energy increases. To be more precise, the first coefficient, $\beta_{0}$, is given in QCD by

$$
\begin{equation*}
\beta_{0}=\frac{11 C_{A}-4 n_{f} T_{F}}{12 \pi} \tag{1.20}
\end{equation*}
$$

where $C_{A}$ and $T_{F}$ are group-related constant and $n_{f}$ is the number of flavour in the theory. For $S U(3)$ and for $n_{f}<17$ the beta function is negative. Solving Eq.(1.18) at lowest order we get

$$
\begin{equation*}
\alpha_{s}\left(Q^{2}\right)=\frac{\alpha_{s}\left(\mu^{2}\right)}{1+\beta_{0} \alpha_{s}\left(\mu^{2}\right) \log \left(Q^{2} / \mu^{2}\right)} \tag{1.21}
\end{equation*}
$$

As $Q^{2}$ approaches infinity, $\alpha_{s}$ vanishes. This property is called Asymptotic Freedom.

Here is the solution to our dilemma: it may be possible to apply perturbation theory in QCD, provided we only consider processes at sufficiently high energies. Nonetheless, we cannot avoid to deal with strongly interacting initial and final states, namely hadrons. We will give the solution to this issue in the next section.

We conclude with an important remark about the running coupling. There is an energy scale at which Eq.(1.21) becomes bigger than unity and
eventually diverges. This energy scale is usually written as $\Lambda_{Q C D}$ and it is called the landau pole of the quantum theory. It is not physical, since perturbation theory has already broken down ${ }^{8}$ at a slight bigger energy (or smaller energy, in QED), when $\alpha_{s} \approx 1$. Below (Above) that threshold, QCD (QED) phenomena are inevitably non perturbative. That is the region where confinement takes place and quarks and gluons bound themselves in hadronic matter. We are now ready to explain how we can describe hadrons.

### 1.3 The parton model

We are interested in processes involving hadrons in initial and final states. A first model was proposed by Feynman and it is usually referred to as the Naive Parton Model. Though this is the old approach to the matter, for pedagogical reasons we present it here in its original formulation. The starting point is the assumption that even if we are not able to describe hadrons in a non-perturbative way, nonetheless they are made up of elementary partons. Consequently, in order to compute hard scattering process, we can try to describe this initial state in terms of elementary partons. As first example, consider a process in which an electron exchange a photon of high virtuality with a proton and produces hadrons as final states. This process is schematically depicted in Figure 1.1a and it is called DIS (Deep Inelastic Scattering). It has been studied both, theoretically and experimentally, since the 1960s. Although it is the simplest process involving hadron in the initial state, yet its description gives a complete overview of the techniques used in more intricate situations.

In the parton model we assume that each parton carries a fraction $z_{i}$ of the total momentum $p$ of the proton, that is

$$
\begin{align*}
& p=\sum_{i} z_{i} p=\sum_{i} \hat{p}_{i} \\
& 0 \leq z_{i} \leq 1 \quad \text { and } \quad \sum_{i} z_{i}=1 \tag{1.22}
\end{align*}
$$

Also, each parton is perfectly collinear to the hadron it comes from. Moreover, we need to neglect parton masses, otherwise Eq.(1.22) makes no sense. We define $f_{i}^{p}(z, i)$ to be the probability density of extracting a parton $i=q_{f}, g, \gamma, \ldots$, with momentum fraction $z_{i}$, from the proton. These Parton Distribution Functions, PDFs, contain all the non-perturbative information.

[^5]

Figure 1.1: A graphic representation of DIS and the parton model

PDFs cannot be calculated perturbatively and need to be extracted from data ${ }^{9}$.

Then, in order to calculate the cross section of the process, we assume the following decomposition to hold

$$
\begin{equation*}
\sigma(p)=\sum_{i} \int_{0}^{1} d z f_{i}(z) \hat{\sigma}(z p) \tag{1.23}
\end{equation*}
$$

where $\sigma$ is the cross section for the process we are interested in and $\hat{\sigma}$ is the cross section for the partonic process. In our example the process is $e+p \longrightarrow e+X$ and the partonic cross section refers to the process $e+p_{i} \longrightarrow e+X$.
Whereas the former is not calculable in perturbative-QCD, on the contrary the latter is, provided the energy scale of the partonic process lies in the perturbative region of the running coupling.
Some considerations are necessary:

- If we want our model to be predictive, PDFs must be process independent. Once they have been fitted, it must be possible to use them in any other process involving the same initial hadron ${ }^{10}$.
- PDFs obey sum rules. For a proton, we require that

$$
\begin{equation*}
\int_{0}^{1} d z\left[f_{u}(z)-f_{\bar{u}}(z)\right]=2 \quad \text { and } \quad \int_{0}^{1} d z\left[f_{d}(z)-f_{\bar{d}}(z)\right]=1 \tag{1.24}
\end{equation*}
$$

[^6]which reproduce the description of a proton as $p=u u d$. The conservation of total longitudinal momentum holds as well
\[

$$
\begin{equation*}
\sum_{i} \int_{0}^{1} d z z f_{i}(z)=1 \tag{1.25}
\end{equation*}
$$

\]

We shall address the problem of the survival of the parton model after QCD corrections have been added. In the next section we will see how QCD corrections lead to an improved formulation of the parton model (which reduces to the original formulation at Leading Order).

### 1.4 QCD corrections to DIS

In this section we retain DIS as example and look into QCD correction to the Leading Order calculation. We also have to explain in more detail what Leading Order means in QCD calculations. DIS is an electromagnetic initiated process, so it starts with a cross section proportional to $\alpha_{e m}^{2}$. The Leading Order (LO) contains no power of $\alpha_{s}$. What we add at Next to Leading Order (NLO) are not electromagnetic corrections, rather we consider QCD corrections, being $\alpha_{s} \gg \alpha_{e m}$. In general, it is not obvious a priori what is a LO calculation in term of powers of the QCD coupling. It depends on the process considered. For DIS, as well as for Drell-Yan (see next chapter), the LO has no power of $\alpha_{s}$. On the contrary, for Higgs production, the LO is proportional to $\alpha_{s}^{4}$. For a generic process $X$, wi write

$$
\begin{equation*}
\sigma_{X}=\sigma_{X}^{L O} \alpha_{s}^{k}+\sigma_{X}^{N L O} \alpha_{s}^{k+1}+\mathcal{O}\left(\alpha_{s}^{k+2}\right) . \tag{1.26}
\end{equation*}
$$

We are ready to consider QCD corrections for DIS. In the naive parton model a quark is extracted from a proton and it scatters, via photon exchange, with the incoming lepton. Usually, however, both the incoming quark and the outgoing one radiate one or more gluons. Real emission corrections of order $\mathcal{O}\left(\alpha_{s}\right)$ are shown in Figures 1.2a, 1.2b.

Moreover, virtual diagram should be taken into account. In particular, the self energy diagram for both quark lines and the photon vertex correction, Figure. 1.2c, 1.2d.

Once these terms have been included, the calculation gives rise to two classes of divergences:


Figure 1.2: QCD correction to DIS

- Ultraviolet (UV) divergences in loop integrals, as we pointed out in Section 1.2.
- Infrared and Collinear (IR) divergences in both, loop integrals and phase space integrals for real emissions.

We have already explained how UV divergences are controlled by renormalization. On the contrary, IR divergences are the big novelty. First of all we should say where they come from. Consider the diagram in Figure 1.2b and let the four momenta $r$ of the emitted gluon approaching zero. According to Feynman rules, a propagator with momentum $k$ is associated to the virtual quark

$$
\begin{equation*}
\frac{i(\not k-m)}{k^{2}-m^{2}+i \epsilon} . \tag{1.27}
\end{equation*}
$$

The denominator Eq.(1.27) vanishes when the on-shellness condition is satisfied, that is when the particle's virtuality reduces to zero as it was a real one. A vanishing momentum $r$ realizes such condition, in fact

$$
\begin{equation*}
l=k+r \longrightarrow k \quad \text { as } \quad r \rightarrow 0, \tag{1.28}
\end{equation*}
$$

and $l$ is on-shell, being the outgoing quark a real particle. The propagator blows up when the emitted gluon carries a very small energy. We will refer
to radiated partons with vanishing energy as soft partons. There is more: we are considering high energy processes in which all masses may be neglected. In a massless theory, the emitted particle may become collinear to the emitting one, that is it has vanishing transverse momentum ${ }^{11}$.
In this case we have

$$
\begin{equation*}
r=z k, \quad l=k-r=(1-z) k, \tag{1.29}
\end{equation*}
$$

$z$ being the fraction of momenta carried by the gluon. In both cases, Eqs. (1.28,1.29),

$$
\begin{equation*}
k^{2} \longrightarrow m^{2}=0, \tag{1.30}
\end{equation*}
$$

and the propagator diverges.
An analogous situation can be found in loop diagram integrals.
For some of these divergences, the machinery of perturbation theory fixes itself. The Kinoshita - Lee - Nauenberg Theorem ensures the soft finiteness of any theory in the Standard Model, see [18, 19]. It states indeed that soft singularities always cancel between real and virtual contribution.
On the other hand, for what concerns collinear divergences, only a partial cancellation takes place and we have to deal with them.
The solution to this problem its very similar to the renormalization procedure. In the Naive Parton Model we consider PDFs in order to take into account the non-perturbative physics. We can think to those distribution functions as bare objects, in analogy to the bare parameter in the lagrangian. Then, we redefine PDFs reabsorbing collinear divergences into them. Just like renormalization procedure, an arbitrary scale $\mu_{F}$, known as the factorization scale, is introduced through the procedure.
We write, instead of Eq.(1.23), the following Improved Parton Model formula:

$$
\begin{equation*}
\sigma(p)=\sum_{i} \int_{0}^{1} d z f_{i}\left(z, \mu_{F}^{2}\right) \hat{\sigma}\left(z p ; \mu_{F}^{2}, \mu_{R}^{2}, \alpha_{s}\left(\mu_{R}^{2}\right)\right) \tag{1.31}
\end{equation*}
$$

This procedure need to fulfill two important properties:

- In any perturbative calculation, we need collinear divergences to factorize exactly, otherwise we can not absorb them consistently;

[^7]- In Section 1.3 we pointed out the necessity for PDFs to be process independent. In order for this requirement to hold after collinear contribution have been absorbed, it is necessary for such contributions to be universal.

Theorems known as Collinear Factorization Theorems ensure that this two facts are true, at least for a large class of processes.

There is one last thing to discuss about improved PDFs, which is their running with energy. Any observable must be independent from the factorization scale choice. This allows us to find a set of evolution equation for the distribution functions. We will discuss this in the next section.

### 1.5 DGLAP evolution equations

It has been briefly anticipated at the end of the previous section, that the $\mu_{F}$ dependence of PDFs is fixed imposing invariance of observable against it. Moreover, the universality of distribution functions implies that also their evolution should be universal, that is process independent. In close analogy to Eq.(1.17), we impose

$$
\begin{equation*}
\frac{1}{\mu_{F}^{2}} \frac{\partial}{\partial \mu_{F}^{2}} \mathcal{O}=0, \tag{1.32}
\end{equation*}
$$

for a generic observable $\mathcal{O}$.
A set of $N_{f}+1$ differential equations satisfies the request. These equations are universally known as Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations (DGLAP), independently found in 1972 by Gribov and Lipatov [17], and than again in 1977 by Dokshitzer [11], and by Altarelli and Parisi [1]:

$$
\begin{gather*}
\frac{1}{\mu_{F}^{2}} \frac{\partial}{\partial \mu_{F}^{2}}\binom{f_{q_{i}}\left(x, \mu_{F}^{2}\right)}{f_{g}\left(x, \mu_{F}^{2}\right)}= \\
\int_{x}^{1} \frac{d \xi}{\xi}\left(\begin{array}{cc}
P_{q_{i}, q_{j}}\left(\frac{x}{\xi}, \alpha_{s}\left(\mu_{F}^{2}\right)\right) & P_{q_{i}, g}\left(\frac{x}{\xi}, \alpha_{s}\left(\mu_{F}^{2}\right)\right) \\
P_{g, q_{j}}\left(\frac{\xi}{\xi}, \alpha_{s}\left(\mu_{F}^{2}\right)\right) & P_{g, g}\left(\frac{\xi}{\xi}, \alpha_{s}\left(\mu_{F}^{2}\right)\right)
\end{array}\right)\binom{f_{q_{j}}\left(\xi, \mu_{F}^{2}\right)}{f_{g}\left(\xi, \mu_{F}^{2}\right)}, \tag{1.33}
\end{gather*}
$$

where $P_{a, b}$ are known asAltarelli-Parisi(AP) splitting functions, to be understood as the probability of a parton $a$ emitting collinear stuff and resulting in a parton $b$. The splitting functions are universal and can be computed perturbatively as

$$
\begin{equation*}
P_{a, b}\left(x, \alpha_{s}\right)=\alpha_{s} P_{a, b}^{(0)}(x)+\alpha_{s}^{2} P_{a, b}^{(1)}(x)+\alpha_{s}^{3} P_{a, b}^{(2)}(x)+\ldots \tag{1.34}
\end{equation*}
$$

As an example, we consider the $P_{g g}$ which is given, at lowest order, by

$$
\begin{equation*}
P_{g g}(x)=2 C_{A}\left[\frac{x}{(1-x)_{+}}+\frac{1-x}{x}+x(1-x)\right]+\beta_{0} \delta(1-x) \tag{1.35}
\end{equation*}
$$

where $\beta_{0}$ is the first coefficient of the beta function, Eq.(1.18). Apart from the customary delta distribution, there is a term proportional to a plus prescription. The definition of plus distributions and their properties can be found in Appendix [A.2]

Finally, splitting functions satisfy numerous constraints due to symmetries. For example, charge conjugation invariance and $S U\left(N_{f}\right)$ flavour symmetry imply that

$$
\begin{equation*}
P_{q_{i}, q_{j}}=P_{\bar{q}_{i}, \bar{q}_{j}}, \quad P_{q_{i}, \bar{q}_{j}}=P_{\bar{q}_{i}, q_{j}}, \quad P_{g, q_{i}}=P_{g, \bar{q}_{i}}, \tag{1.36}
\end{equation*}
$$

and they are also flavour independent.
Solution of the DGLAP equations corresponds to the resummation of collinear logs up to a desired scale $Q^{2}$. The DGLAP equations and their solution are not indispensable topics for this thesis. Nonetheless, there is one last rehashing we want to show which will appear again in what follows. First, we note that Equations. (1.33) are written in the form of a convolution, that is

$$
\begin{equation*}
\frac{\partial}{\partial \log \mu^{2}} f\left(x, \mu^{2}\right)=\int_{x}^{1} \frac{d \xi}{\xi} P\left(\frac{x}{\xi}, \mu^{2}\right) f\left(\xi, \mu^{2}\right) \equiv P \otimes f\left(x, \mu^{2}\right) . \tag{1.37}
\end{equation*}
$$

This integral product factorizes under a particular integral transform, the Mellin Transform, defined in Appendix. [A.1]. Computing the Mellin transform on both side, we get

$$
\begin{equation*}
\frac{\partial}{\partial \log \left(\mu^{2}\right)} \tilde{f}\left(N, \mu^{2}\right)=\tilde{P}\left(N, \alpha_{s}\left(\mu^{2}\right)\right) \tilde{f}\left(N, \alpha_{s}\left(\mu^{2}\right)\right) \tag{1.38}
\end{equation*}
$$

In this particular case, the Mellin transform has reduced the problem of solving a integro-differential equation into the simpler task of solving differential equation of the first order.

### 1.6 Hadron collisions

Although DIS would be sufficient to proceed with general resummation theory, it is however useful to extend the (improved) parton model to a generic
process involving two hadrons in the initial state. We have seen how cross sections are computed in perturbative-QCD (pQCD). We explicitly considered DIS as an example, giving Eq.(1.31) as main formula. We want to extend (1.31) to the more difficult case of hadron collisions. The practical reason is obvious, being LHC a hadron collider. In turn, the phenomenological interest arises from the necessity of assessing QCD as the fundamental theory of strong interactions.
Two processes have been widely studied, both theoretically

- The Drell-Yan process: $p p \longrightarrow \gamma^{*} / Z^{*} \longrightarrow l \bar{l}+X$. A lepton pair is produced in proton-proton $p p$ collision with a gauge boson as intermediate state. This is a benchmark for the study of gauge bosons, and it has been a fundamental step in experimentally assessing QCD as a whole ${ }^{12}$.
- Higgs production: $p p \longrightarrow H+X$. In this case the object of interest is the Higgs boson and its coupling to quarks. Reasons for studying this process are due to the interest in the Higgs boson itself.

Both processes are well suited for the study of resummation but, for simplicity, in this thesis we will only consider Higgs production.
The generalization of Eq.(1.31) is formally straightforward. Instead of a single parton distribution function, we consider a pair, one for each incoming hadron, and we integrate them with a partonic cross section. If we indicate with $S$ the final state (lepton pairs, Higgs, ...) and with $Q^{2}$ its invariant mass, then the master formula for the inclusive cross section is readily given

$$
\begin{equation*}
\sigma_{S}=\sum_{a, b} \int_{0}^{1} d x_{1} d x_{2} f_{a}\left(x_{1} ; \mu_{F}^{2}\right) f_{b}\left(x_{2} ; \mu_{F}^{2}\right) \times \hat{\sigma}_{a b \rightarrow S}\left(x_{1}, x_{2} ; \alpha_{s}\left(\mu_{R}^{2}\right), \frac{Q^{2}}{\mu_{F}^{2}}, \frac{Q^{2}}{\mu_{R}^{2}}\right) \tag{1.39}
\end{equation*}
$$

where $x_{1}$ and $x_{2}$ are the momentum fractions carried by parton $a$ and $b$, respectively. The dependence on factorization and renormalization scales is explicitly highlighted. If we wanted to consider less inclusive observable, such as the single differential distribution or a double differential distribution, we would have other dependencies: transverse momentum, rapidity, ... We will consider these cases later in this thesis.

[^8]The structure of Eq.(1.39) is very similar to Eq.(1.31), namely it is made of two parts, that we recall here: PDFs embodies the non perturbative information about particle contents of hadrons ${ }^{13}$, whereas the partonic cross section, $\hat{\sigma}$, is given as perturbative series in the coupling.
We note that different sub-processes, that is $\hat{\sigma}_{a b \rightarrow S}$ as $a, b$ change, contribute with the different weights to the final result. For Drell-Yan, the main channel is quark anti-quark annihilation, $q \bar{q} \rightarrow l \bar{l}+X$. In the case of Higgs production, the gluon-fusion initiated process is the dominant one, $g g \rightarrow H+X$.

We now introduce some standard notation and recast Eq.(1.39) in a different fashion. The following manipulations are strictly suitable for completely inclusive cross section. In the case of semi-inclusive or completely differential cross sections, a few adjustments will be necessary.
We define the variable $\tau$ as

$$
\begin{equation*}
\tau=\frac{Q^{2}}{s} \tag{1.40}
\end{equation*}
$$

where $s$ is the customary Mandelstam invariant for the centre of mass energy at hadrons level and $Q^{2}$ is the invariant mass of the observed final state: $Q^{2}=M_{H}^{2}$ in the case of Higgs production. The partonic counterpart $z$ of $\tau$ is defined as

$$
\begin{equation*}
z=\frac{Q^{2}}{\hat{s}}=\frac{\tau}{x_{1} x_{2}} \tag{1.41}
\end{equation*}
$$

being $\hat{s}$ the partonic centre of mass energy. In term of these variables, the inclusive cross section is given by

$$
\begin{align*}
\frac{1}{\tau} \sigma^{H}(\tau) & =\sum_{i j} \int_{\tau}^{1} \frac{d z}{z} \int_{\frac{\tau}{z}}^{1} \frac{d y}{y} f_{i}^{(1)}\left(\frac{\tau}{z y}\right) f_{j}^{(2)}(y) \hat{\sigma}_{i j}^{H}(z) \\
& =\sum_{i j} \int_{\tau}^{1} \frac{d z}{z} \mathcal{L}_{i j}\left(\frac{\tau}{z}\right) C_{i j}(z), \tag{1.42}
\end{align*}
$$

having defined the Parton Luminosity

$$
\begin{equation*}
\mathcal{L}_{i j}(x)=c_{i j} \int_{x}^{1} \frac{d y}{y} f_{i}^{(1)}\left(\frac{x}{y}\right) f_{j}^{(2)}(y) . \tag{1.43}
\end{equation*}
$$

The $C_{i j}$ coefficients are usually referred to as coefficient functions. It is just a matter of convenience to decide on normalizing the perturbative series in such a way that $C_{i j}^{(0)}=\delta(1-z)$, absorbing all numerical prefactors into the $c_{i j}$ coefficients which appear into the luminosity definition. From

[^9]the definitions of $\tau$ and $z$ follows that the threshold region is identified by $\tau \rightarrow 1$ or, at parton level, by $z \rightarrow 1$.

We will consider less inclusive observable in the next Chapter. For now, Eq. (1.42) is the result to keep in mind.

## Chapter 2

## Resummation theory

In this chapter we briefly review the general theory of resummation. We want to explain the motivations and the general approach. We first explain the origin of soft large logarithms. Then, we build a very simplified argument for resummation and we give without proof the general results for inclusive distributions and for single differential cross sections. Finally, we present a check of validity of one of the resummation formulae given.

### 2.1 Large Logarithms and Resummation

We consider from a general point of view the problem of large logarithms and their resummation. We point out the origin of such soft enhanced terms and explain the big idea behind resummation procedure.

### 2.1.1 Soft Large Logarithms: the origin

In order to understand where large logarithms come from, it is necessary to step back and consider again the emission of $n$ gluons by a quark line. The process is schematically depicted in Figure 2.1.

As we have already mentioned, integration over the phase space gives rise to divergent contributions which then cancel against loop integrals. When the emission of $n$ gluons is considered, the result of such a cancellation is a term that looks like

$$
\begin{equation*}
\alpha_{s}^{n}\left(\frac{\log ^{k}(1-z)}{1-z}\right)_{+}, \quad 0 \leq k \leq a n-1, \tag{2.1}
\end{equation*}
$$



Figure 2.1: Emission of n gluons from a quark parton line. Each gluon carries away a fraction $\left(1-z_{i}\right)$ of the incoming energy. At the end, the quark is left with a fraction $z_{1} \ldots z_{n}$ of its original momentum.
where $a=1$ for DIS and $a=2$ for Drell-Yan and Higgs.
It is now clear when resummation is needed: roughly, for a value of $x$ such that

$$
\begin{equation*}
\alpha_{s} \log ^{2}(1-x) \sim 1, \tag{2.2}
\end{equation*}
$$

due to the fact that contributions of the form of Eq. (2.1) arise at any perturbative order, all terms in the perturbative series become of the same order and perturbation theory is no more reliable. We need to "resum" these contributions.

We anticipate here that, very roughly speaking, resummation theory provides a sort of reorganization of part of the perturbative series in powers of $\alpha_{s} \log ^{k}\left(\alpha_{s}\right)$, for values of $k$ from 0 to some process-dependent maximum value ( $k_{\max }=2$ for Higgs production and Drell-Yan), followed by their exponentiation.

### 2.1.2 Soft Large Logarithms: resummation

Resummation of inclusive cross section has been achieved long ago, see [25, 9 ], and more recently using renormalization group arguments [15]. We just want to outline here the procedure and main results.

Resummation relies on two fundamental facts: factorization in the soft limit of both, the matrix element and the phase space, and exponentiation of the factorized contributions.
For what concerns the matrix element factorization it can be proved that, if $M_{n}$ is the amplitude which includes the emission of $n$ gluons, in the soft
limit it reduces to

$$
\begin{equation*}
M_{n}\left(z_{1}, \ldots, z_{n}\right) \approx\left[\frac{1}{n!} \prod_{i=1}^{n} M_{1}\left(z_{i}\right)\right] \times M_{0} \tag{2.3}
\end{equation*}
$$

where $z_{i}$ are the vanishing fraction of energy of the emitted gluons and $M_{0}$ is the amplitude with no extra gluons. In QED the proof is quite simple, whereas in QCD complications are due to the non abelian nature of strong interactions.
On the contrary, in the same kinematic configuration, phase space does not factorize. It gives instead a constraints of the form

$$
\begin{equation*}
d z_{1} d z_{2} \cdots d z_{n} \delta\left(z-z_{1} z_{2} \cdots z_{n}\right) \tag{2.4}
\end{equation*}
$$

which is clearly not factorized. Nonetheless, we observe that under a Mellin transform phase space factorizes too. In fact, we have

$$
\begin{equation*}
\int_{0}^{1} \frac{d z}{z} z^{N} \delta\left(z-z_{1} z_{2} \cdots z_{n}\right)=z_{1}^{N} z_{2}^{N} \cdots z_{n}^{N} \tag{2.5}
\end{equation*}
$$

Overall, factorization does actually take place, provided we consider it in the reciprocal Mellin space.

We also observe that Eqs. $(1.42,1.43)$ are in the form of a double multiplicative convolution. Then, if we take the $N$-th moment on both side of Eq. (1.42), we get

$$
\begin{align*}
\tilde{\sigma}^{H}(N) & =\sum_{i j} \tilde{\mathcal{L}}_{i j}(N) \tilde{C}_{i j}(N) \\
& =\sum_{i j} \tilde{f}_{i}^{(1}(N) \tilde{f}_{j}^{(2}(N) \tilde{C}_{i j}(N) \tag{2.6}
\end{align*}
$$

In $N$-space it is also simple to isolate threshold contributions. In fact, in the large $N$ limit, only the region $z \approx 1$ is picked up. We refer to Appendix [A.3] for a more quantitative explanation. In $N$-space with large $N$, the tower of logarithms of Eq. (2.1) is converted into the tower of logarithms

$$
\begin{equation*}
\alpha_{s}^{n} \log ^{k}(N) \quad 0 \leq k \leq 2 n . \tag{2.7}
\end{equation*}
$$

In general, a plus prescription of the form of Eq. (2.1) in direct space converts into a complicated function in $N$-space, which behaves, for large N , as $\log ^{k+1}(N)$. Regular pieces in the partonic cross section are suppressed in the large $N$ limit by some power of $N$. More details about the the behaviour
of function and distribution under the Mellin transform are summarized in Appendix [A.3].

Factorization of both, the matrix element and phase space, results in the complete factorization of the coefficient function itself:

$$
\begin{equation*}
C_{n}(N)=\frac{1}{n!}\left[C_{1}(N)\right]^{n}, \tag{2.8}
\end{equation*}
$$

where $C_{n}$ is a shorthand for the coefficient function due to the emission of $n$ gluons. Retaining only the most divergent (leading) term, we would find

$$
\begin{align*}
C_{1}^{\text {soft }}(N) & \stackrel{N \geqq}{=} \int_{0}^{1} d z z^{N-1} 4 A_{1}\left(\frac{\log (1-z)}{1-z}\right)_{+}  \tag{2.9}\\
& =2 A_{1} \log ^{2}\left(\frac{1}{N}\right)+\mathcal{O}\left(\frac{1}{N}\right),
\end{align*}
$$

where $A_{1}=C_{A} / \pi$ for Higgs. Resummation of Leading Logs (LL) is almost done. If we sum over all possible number of gluons we get the promised exponentiation

$$
\begin{gather*}
C^{\mathrm{res}}\left(N, \alpha_{s}\right)=\sum_{n=0}^{\infty} \alpha_{s}^{n}\left[C_{n}(N)\right]_{\mathrm{soft}}  \tag{2.10}\\
=\sum_{n=0}^{\infty} \frac{\alpha_{s}^{n}}{n!}\left[C_{n}(N)\right]_{\text {soft }}^{n}=\exp \left[\alpha_{s} C_{1}^{\mathrm{soft}}(N)\right] .
\end{gather*}
$$

### 2.2 General Resummation Theory

The simple discussion above given can be put in a more complete form, also taking into account effects due to the running of the coupling. We will only give the general result with no proof.
The right expression to substitute Eq. (2.10) with, is, see [9, 25, 15],

$$
\begin{equation*}
C^{\text {res }}\left(N, \alpha_{s}\right)=g_{0}\left(\alpha_{s}\right) \exp \mathcal{S}\left(\alpha_{s} \log (N), \alpha_{s}\right) \tag{2.11}
\end{equation*}
$$

where $g_{0}$ is given as a power series in the coupling and it contains all the constant terms ${ }^{1}$ and the exponent $\mathcal{S}\left(\lambda, \alpha_{s}\right)$ is called Sudakov form factor

[^10]and it collects all the logarithmically enhanced contributions. Its logarithmic expansion is given by
\[

$$
\begin{equation*}
\mathcal{S}\left(\lambda, \alpha_{s}\right)=\frac{1}{\alpha_{s}} g_{1}(\lambda)+g_{2}(\lambda)+\alpha_{s} g_{3}(\lambda)+\cdots, \tag{2.12}
\end{equation*}
$$

\]

where $\lambda=\alpha_{s} \log (N)$.
Formal expansions of the $g$ coefficients are

$$
\begin{gather*}
g_{0}\left(\alpha_{s}\right)=\sum_{j=0}^{\infty} g_{0 j} \alpha_{s}^{j}  \tag{2.13}\\
g_{k}(\lambda)=\sum_{j=1}^{\infty} g_{k j} \lambda^{j}, \quad g_{11}=0, \tag{2.14}
\end{gather*}
$$

In order to better understand what $\mathrm{N}^{k}$ LL accuracy means, we first expand Eq. (2.11) up to some fixed order, for example $\mathcal{O}\left(\alpha_{s}^{2}\right)$.
Defining $L=\log (N)$,

$$
\begin{align*}
C^{\text {res }}\left(N, \alpha_{s}\right) & =g_{00}+\alpha_{s}\left[g_{12} L^{2}+g_{21} L+g_{01}\right] \\
& +\alpha_{s}^{2}\left[\frac{g_{12}^{2}}{2} L^{4}+\left(g_{12} g_{21}+g_{13}\right) L^{3}+\left(\frac{g_{21}^{2}}{2}+g_{22}+g_{12} g_{02}\right) L^{2}+\mathcal{O}(L)\right] \\
& +\mathcal{O}\left(\alpha_{s}^{3}\right) . \tag{2.15}
\end{align*}
$$

The LL accuracy is given by the function $g_{1}$ and it predicts the complete series of logarithms ${ }^{2} \alpha_{s}^{n} L^{2 n}$. If we included the function $g_{2}$, then we would have a NLL prediction. At NLL accuracy towers of logarithm $\alpha_{s}^{n} L^{k}$ with $2 n-2 \leq k \leq 2 n$ are included.
We observe that in the exponential expansion, also logarithms of lower order appear. It is important to point out that their coefficient cannot be correctly predicted at this accuracy. To better understand this fact, consider, for example, Eq. (2.15). It is the correct formal expansion for a NLL prediction, written as a series up to NNLO. Without the inclusion of the $g_{3}$ function, a contribution of order $\alpha_{s}^{2} L$ is naturally missing. We could incorrectly - say that the coefficient of the term $\alpha_{s}^{2} L$ is zero. This is true only at NLL accuracy. Moreover, if we computed the series expansion up to $\mathcal{O}\left(\alpha_{s}^{3}\right)$, even if we are not considering the $g_{3}$, would found terms out of the

[^11]range $2 n-2 \leq k \leq 2 n$ due to interference of lower order $g_{i}$ functions. Again, also their numerical coefficients are wrong. All these logarithms belong to the NNLL accuracy, which need the inclusion of $g_{3}$.

In general, a $\mathrm{N}^{p} \mathrm{LL}$ prediction need the inclusion of the $g_{i}$ function up to $i=k+1$, and the correct expansion of the $g_{0}$ to the same perturbative order. Terms up to the $k$-th power of $\ln (N)$, with $2(n-p) \leq k \leq 2 n$ will be correctly predicted.

A remark about the $g_{0}$ : though in a LL prediction $g_{0}$ is completely harmless, this is not true for a higher logarithmic accuracy. It is clear, from Eq. (2.15), that the $g_{0}$ can interfere with the $g_{i}$, giving important logarithmic terms.

## The matching procedure

We conclude this section with a brief description of what is called the matching procedure. In fact, the resummed expression need to be matched with the fixed order prediction. The former, is a valid improvement only near the threshold, whereas the latter is the correct perturbative approximation far from the threshold region. We can't just add together the resummed expression and the fixed order one, because we would count twice large logarithms coming from low perturbative order. The right thing to do is to add the two pieces but also subtract the fixed order expansion of the resummed expression. This is more easily understandable in formulae

$$
\begin{equation*}
C_{N^{k} L L}^{N^{p} L O}\left(N, \alpha_{s}\right)=\sum_{j=0}^{p} \alpha_{s}^{j} C^{j}(N)+C_{N^{k} L L}^{r e s}\left(N, \alpha_{s}\right)-\sum_{j=0}^{p} \frac{\alpha_{s}^{j}}{j!}\left[\frac{d^{j} C_{N^{k} L L}^{r e s}\left(N, \alpha_{s}\right)}{d \alpha_{s}^{j}}\right] . \tag{2.16}
\end{equation*}
$$

Eq. (2.16), or rather its inverse Mellin transform, is the final observable result.
We want now to extend this treatment to less inclusive quantities, in particular differential distribution.

### 2.3 Transverse Momentum Distribution

In this section we consider transverse momentum distribution for the production of a colour singlet system

$$
\begin{equation*}
h_{1}+h_{2} \longrightarrow \mathcal{S}+X . \tag{2.17}
\end{equation*}
$$

We indicate with $p_{T}$ the transverse momentum of the target system $\mathcal{S}$ and with $Q^{2}$ its invariant mass, whereas $M_{X}^{2}$ will be invariant mass of the extra radiation.
Requiring the invariant mass of extra radiations to be non negative, $M_{X}^{2} \geq 0$, then the integral in Eq.(1.39) is restricted to

$$
\begin{align*}
& \frac{d \sigma_{S}}{d \xi_{p}}\left(\tau, \xi_{p} ; \alpha_{s}\left(\mu_{R}^{2}\right), \mu_{R}^{2}, \mu_{F}^{2}\right)=\sum_{i j} \int_{\tau\left(\sqrt{\xi_{p}+1}+\sqrt{\xi_{p}}\right)^{2}}^{1} d x_{1} f_{i}\left(x_{1}, \mu_{F}^{2}\right) \\
& \frac{\int_{\tau}^{1}\left(\sqrt{\xi_{p}+1}+\sqrt{\xi_{p}}\right)^{2}}{x_{1}} d x_{2} f_{j}\left(x_{2}, \mu_{F}^{2}\right) \frac{d \hat{\sigma}_{i j}}{d \xi_{p}}\left(\frac{\tau}{x_{1} x_{2}} ; \alpha_{s}, \mu_{R}^{2}, \mu_{F}^{2}\right), \tag{2.18}
\end{align*}
$$

where $\xi_{p} \equiv \frac{p_{T}^{2}}{Q^{2}}$. For later convenience, we define the parameter

$$
\begin{equation*}
a\left(\xi_{p}\right):=\left(\sqrt{\xi_{p}+1}+\sqrt{\xi_{p}}\right)^{2} \tag{2.19}
\end{equation*}
$$

We are interested in finding a transformation which recast Eq.(2.18) in the form of a convolution, as Eq.(1.42) was. That is provided by the following rescaling of the variable $\tau$, Eq. (1.40), and, consequently, $z$, Eq. (1.41)

$$
\begin{equation*}
\tau^{\prime}:=\tau a\left(\xi_{p}\right) \quad z^{\prime}=z a\left(\xi_{p}\right) \tag{2.20}
\end{equation*}
$$

After the rescaling, we get

$$
\begin{align*}
\frac{1}{\tau^{\prime}} \frac{d \sigma}{d \xi_{p}}\left(\tau^{\prime}\right) & =\sum_{i j} \int_{\tau^{\prime}}^{1} \frac{d z^{\prime}}{z^{\prime}} \int_{\frac{\tau^{\prime}}{z^{\prime}}}^{1} \frac{d y}{y} f_{i}^{(1)}\left(\frac{\tau^{\prime}}{z^{\prime} y}\right) f_{j}^{(2)}(y) \frac{d \hat{\sigma}_{i j}}{d \xi_{p}}\left(z^{\prime}\right) \\
& =\sum_{i j} \int_{\tau^{\prime}}^{1} \frac{d z^{\prime}}{z^{\prime}} \mathcal{L}_{i j}\left(\frac{\tau^{\prime}}{z^{\prime}}\right) C_{i j}\left(z^{\prime}\right) \tag{2.21}
\end{align*}
$$

This equation has precisely the same structure of Eq. (1.42).
Before discussing threshold resummation, it is worth to mention that a different kind of resummation is also needed for $p_{T}$-differential distribution in some kinematic region. In fact, when the transverse momentum of the final state approaches zero, one finds that the fixed order cross section is actually divergent (beyond the LO). In that region a different kind of resummation is needed: transverse momentum or collinear resummation. We are not going to investigate transverse momentum resummation and we refer
the interested reader to the literature: see, for example, $[10,8]$ and many others. We just mention here that, in order to perform transverse momentum resummation, a two dimensional Fourier transform is the right recipe. The Fourier transform is taken with respect to the transverse momentum. In Fourier $b$-space the region of small $p_{T}$ is converted in the limit $b \rightarrow \infty$.

We now turn to threshold resummation at fixed $p_{T}$. It was first derived at NLL in $[14,6]$. Using renormalization group arguments, $[15,5]$, it is possible to write a formal decomposition for the partonic cross section. Following the notation of Ref. [21] ${ }^{3}$,

$$
\begin{equation*}
\frac{d \hat{\sigma}_{i j}}{d \xi_{p}}=\sigma_{0}\left(C_{0}\left(N, \xi_{p}\right)\right)_{i j}\left(g_{0}\right)_{i j}\left(\xi_{p}\right) \exp \left[\mathcal{G}\left(N, \xi_{p}\right)\right] \tag{2.22}
\end{equation*}
$$

where the Sudakov-like exponent is now given by

$$
\begin{equation*}
\mathcal{G}\left(N, \xi_{p}\right)=\Delta_{i}(N)+\Delta_{j}(N)+J_{k}(N)+S\left(N, \xi_{p}\right), \tag{2.23}
\end{equation*}
$$

where $k$ labels the recoiling partons. The decomposition in Eq.(2.23) shows the origin of different large logarithms: $\Delta_{i}(N)$ for the incoming partons, $J_{k}(N)$ refers to final recoiling partons and S is a sort of interference factor. Explicitly, the various terms can be written as

$$
\begin{gather*}
\Delta_{i}(N)=\int_{0}^{1} d z \frac{z^{N-1}-1}{1-z} \int_{\bar{Q}^{2}}^{\bar{Q}^{2}(1-z)^{2}} \frac{d q^{2}}{q^{2}} A_{i}^{t h}\left(\alpha_{s}\left(q^{2}\right)\right),  \tag{2.24}\\
J_{k}(N)=\int_{0}^{1} d z \frac{z^{N-1}-1}{1-z} \int_{\bar{Q}^{2}(1-z)^{2}}^{\bar{Q}^{2}(1-z)} \frac{d q^{2}}{q^{2}} A_{k}^{t h}\left(\alpha_{s}\left(q^{2}\right)+B_{k}^{t h}\left(\alpha_{s}\left(\bar{Q}^{2}(1-z)\right)\right),\right. \\
S\left(N, \xi_{p}\right)=-\int_{0}^{1} d z \frac{z^{N-1}-1}{1-z} A_{k}^{t h}\left(\alpha_{s}\left(\bar{Q}^{2}(1-z)^{2}\right)\right) \ln \frac{\left(\sqrt{\xi_{p}+1}+\sqrt{\xi_{p}}\right)^{2}}{\xi_{p}}, \tag{2.25}
\end{gather*}
$$

where $\bar{Q}^{2}=Q^{2}\left(\sqrt{\xi_{p}+1}+\sqrt{\xi_{p}}\right)^{2}$. We observe that, consistently with the rescaling of the relevant dimensionless variables, namely $\tau$ and its partonic counterpart $z$, also the relevant hard scale of the process, namely $Q^{2}$, has been rescaled accordingly.
The functions $A$ and $B$ are given as power series in $\alpha_{s}$. More detail about

[^12]threshold resummation for transverse momentum distribution can be found in the references already given. Resummation results up to NNLL and, in particular, the numerical coefficient of $A$ and $B$ expansions, can be found in ref. [21].

### 2.4 Rapidity Distribution

In this section we consider the case of fixed rapidity. Recall that we denote with $Q$ the four-momentum of the final state,

$$
\begin{equation*}
Q=\left(Q_{0}, \vec{Q}_{T}, Q_{L}\right) \tag{2.27}
\end{equation*}
$$

Let's concentrate on Higgs production, so that $Q^{2}=M_{H}^{2}$.
The rapidity along the third axis of a particle with four momentum $k_{\mu}$, is defined as

$$
\begin{equation*}
Y=\frac{1}{2} \ln \frac{k_{0}+k_{3}}{k_{0}-k_{3}}=\frac{1}{2} \ln \frac{E+k_{z}}{E-k_{z}} . \tag{2.28}
\end{equation*}
$$

The advantage of choosing rapidity instead of longitudinal momentum comes from the facts that it transform additively under Lorentz transformations. This is particularly useful, since we are often interested in switching between the hadronic frame of reference and the partonic one.

We indicate with $y=Y_{H}$ the Higgs rapidity in the hadronic frame of reference. The centre of mass of colliding partons is shifted by an amount which depends on the the values of $x_{1,2}$, the partonic fractions of momentum. Obviously, this shift must vanish when $x_{1}=x_{2}$.
Computing the right Lorentz transformation, it turns out that the rapidity in the partonic frame of reference, $\hat{y}$, is related to the previous one by

$$
\begin{equation*}
\hat{y}=y-\frac{1}{2} \ln \frac{x_{1}}{x_{2}} . \tag{2.29}
\end{equation*}
$$

As promised, the transformation is additive and it vanishes, in this particular situation, when $x_{1}=x_{2}$.

We are ready to give the general formula for the cross section. We skip again the kinematic of the process: we are going to compute it explicitly for
the more intricate case of a double differential distribution. We will highlight there how to recover this result and the one for transverse momentum distribution. If we define

$$
\begin{equation*}
x_{1}^{0} \equiv \sqrt{\tau} e^{y} \quad x_{2}^{0} \equiv \sqrt{\tau} e^{-y}, \tag{2.30}
\end{equation*}
$$

then, the master equation of perturbative QCD , (1.39), can be rewritten with explicit kinematic constraints over $x_{1,2}$ as

$$
\begin{equation*}
\frac{1}{\tau} \frac{d \sigma_{H}}{d y}\left(\tau, y, \alpha_{s}\right)=\sum_{i j} c_{i j} \int_{x_{1}^{0}}^{1} d x_{1} \int_{x_{2}^{0}}^{1} d x_{2} f_{i}\left(x_{1}\right) f_{j}\left(x_{2}\right) C_{i j}\left(z, \tau, \hat{y} ; \alpha_{s}\right), \tag{2.31}
\end{equation*}
$$

with implicit dependence on the factorization and renormalization scales. The hadronic rapidity $y$ and the partonic one $\hat{y}$ are related by Eq. (2.29), whereas $\tau$ and $z$ are defined as in Eqs. (1.40,1.41).
We start by observing that $y$ is restricted by the conditions $x_{1,2} \leq 1$. Requiring $x_{1,2}^{0} \leq 1$, it follows

$$
\begin{equation*}
\frac{1}{2} \ln \tau \leq y \leq \frac{1}{2} \ln \frac{1}{\tau} \tag{2.32}
\end{equation*}
$$

whereas the condition $0 \leq M_{X}^{2}$ computed in the partonic frame of reference brings to the condition

$$
\begin{equation*}
\frac{1}{2} \ln z \leq \hat{y} \leq \frac{1}{2} \ln \frac{1}{z}, \tag{2.33}
\end{equation*}
$$

which is clearly meaningful if the following is satisfied too

$$
\begin{equation*}
z \leq 1 . \tag{2.34}
\end{equation*}
$$

We can recast Eq. (2.31) using Eqs.(2.33,2.34):

$$
\begin{array}{r}
\sigma_{\text {rap }}(\tau, y) \equiv \frac{1}{\tau} \frac{d \sigma_{H}}{d y}\left(\tau, y, \alpha_{s}\right)=\int_{0}^{1} d x_{1} d x_{2} d z \int_{\hat{y}_{\text {min }}}^{\hat{y}_{\text {max }}} d \hat{y} f_{i}\left(x_{1}\right) f_{j}\left(x_{2}\right)  \tag{2.35}\\
\times C(z, \hat{y}) \delta\left(z x_{1} x_{2}-\tau\right) \delta\left(\hat{y}-y+\frac{1}{2} \ln \frac{x_{1}}{x_{2}}\right),
\end{array}
$$

where $\hat{y}_{\text {max }} \equiv \frac{1}{2} \ln \left(\frac{1}{z}\right) \equiv-\hat{y}_{\text {min }}$ and we have suppressed all sub-process indices, for brevity. We define also the hadron-level rapidity extrema $y_{\max } \equiv$ $\frac{1}{2} \ln \left(\frac{1}{\tau}\right) \equiv-y_{\text {min }}$.

It is now clear what we are going to do next. Thanks to the double delta functions, Eq. (2.35) factorizes exactly under a Mellin-Fourier transform, taken with respect to $\tau$ and $y$

$$
\begin{align*}
\sigma_{\text {rap }}(N, M) & \equiv \int_{0}^{1} d \tau \int_{y_{\min }}^{y_{\max }} d y \tau^{N-1} e^{i M y} \sigma_{\text {rap }}(\tau, y)  \tag{2.36}\\
& =f_{1}(N+i M / 2) f_{2}(N-i M / 2) C(N, M),
\end{align*}
$$

where

$$
\begin{equation*}
f_{i}(N \pm i M / 2)=\int_{0}^{1} d x x^{N \pm i \frac{M}{2}-1} f_{i}(x) \tag{2.37}
\end{equation*}
$$

and

$$
\begin{equation*}
C(N, M)=\int_{0}^{1} d z z^{N-1} \int_{\hat{y}_{\text {min }}}^{\hat{y}_{\text {max }}} d \hat{y} e^{\hat{y} i M / 2} C(z, \hat{y}) . \tag{2.38}
\end{equation*}
$$

We will show now that, if we take $M$ fixed, the general resummed expression near threshold is given by the inclusive one plus power suppressed (in N ) corrections.

The coefficient function is symmetric in $\hat{y}$, so we can rewrite Eq. (2.38) considering only the positive domain of integration

$$
\begin{equation*}
C\left(N, M ; \alpha_{s}\right)=2 \int_{0}^{1} d z z^{N-1} \int_{0}^{\hat{y}_{\text {max }}} d \hat{y} \cos (\hat{y} M) C\left(z, \hat{y}, \alpha_{s}\right) . \tag{2.39}
\end{equation*}
$$

One can see that, if $M$ is taken to be finite, in the large $N$ limit only $\hat{y}=0$ gives a contribution. In fact, expanding the cosine

$$
\begin{equation*}
\cos (M y)=1-\frac{M^{2} y^{2}}{2}+\mathcal{O}\left(M^{4} y^{4}\right) \tag{2.40}
\end{equation*}
$$

we see that the first term reproduce the rapidity integrated coefficient function, while the others are power suppressed as $N$ approaches infinity. This can be seen expanding also the upper integration extremum near threshold

$$
\begin{equation*}
\ln \frac{1}{\sqrt{z}}=\frac{1}{2}(1-z)+\mathcal{O}(1-z)^{2} \tag{2.41}
\end{equation*}
$$

and observing that in the integration over $\hat{y}$, starting the second term of the cosine expansion, powers of $\hat{y}$ end up regularizing the integrand.
We can rewrite Eq. (2.36) as

$$
\begin{align*}
\sigma_{\text {rap }}\left(N, M ; \alpha_{s}\right) & =f_{1}(N+i M / 2) f_{2}(N-i M / 2) C\left(N ; \alpha_{s}\right) \times(1+\mathcal{O}(1 / N)) \\
& =\sigma_{r a p}^{r e s}\left(N, M ; \alpha_{s}\right)(1+\mathcal{O}(1 / N)) \tag{2.42}
\end{align*}
$$

where $\sigma_{r a p}^{r e s}\left(N, M ; \alpha_{s}\right)$ is just the inclusive resummed coefficient function times the Mellin-Fourier version of PDFs. We have followed here the derivation given in [5]. More details and phenomenological studies may be found there.

It was proposed years ago, [9], and recently implemented, [4, 3], a different approach, which take into account the full rapidity dependence. In
ref. [5] the threshold limit is taken working in the Mellin-Fourier space, with related variables $(M, N)$, and by taking the limit $N \rightarrow \infty$ while keeping $M$ fixed. As we have just shown, this is equivalent, in the large- $N$ limit, to ignoring rapidity, that is integrate over rapidity. The other possibility is to take a limit on the variable $M$ too. This is easily done working in a double Mellin space, $\left(N_{1}, N_{2}\right)$, which is related to the previous one by the following definitions

$$
\begin{align*}
& N_{1}:=N+i M / 2  \tag{2.43}\\
& N_{2}:=N-i M / 2,
\end{align*}
$$

and then considering the two independent limits $N_{1,2} \rightarrow \infty$. Using the definitions in Eq. (2.43) it is obvious that as $N_{1,2}$ approaches infinity independently the following situations may occur:

- If $N_{1} / N_{2} \rightarrow 1$, then $M$ is kept finite;
- If $N_{1} / N_{2} \rightarrow \rho$ and $\rho \neq 1$, then $M \rightarrow \pm \infty$;

In both situation, since $N=\left(N_{1}+N_{2}\right) / 2$, the threshold limit $N \rightarrow \infty$ is taken. The first case correspond to the threshold limit as it was taken in ref. [5]: Eq. (2.43) implies that $N_{1,2}=N+\mathcal{O}(M / N)$. We are now going to explore the second one.

We have to consider how the second choice changes the integral transform in Eq. (2.36). Obviously, a legitimate choice is performing the Mellin-Fourier transform and only at the end change variables $(N, M) \rightarrow\left(N_{1}, N_{2}\right)$. But, in this case, it is easier to consider from the beginning the change of variables. In fact, as we are going to demonstrate now, in this case it coincides to consider a Mellin-Mellin space instead of a Fourier-Mellin one.
The first observation we make is that in $\left(N_{1}, N_{2}\right)$ space the PDFs depend exactly on those variables separately, Eq. (2.37), and they can be obtained just as Mellin transform, with Mellin variable $N_{i}$. It is less obvious what happens to the coefficient function in $\left(N_{1}, N_{2}\right)$ space. We consider again the integral definition in Eq. (2.38) and perform the following change of variables

$$
\left\{\begin{array}{l}
z=z_{1} z_{2}  \tag{2.44}\\
y=\frac{1}{2} \ln \frac{z_{1}}{z_{2}}
\end{array} \quad \longrightarrow J\left(z_{1}, z_{2}\right)=\left(\begin{array}{cc}
z_{2} & z_{1} \\
\frac{1}{2 z_{1}} & -\frac{1}{2 z_{2}}
\end{array}\right) \quad, \quad|\operatorname{det} J|=1,\right.
$$

so that

$$
\begin{equation*}
z^{N-1} e^{i M \hat{y}}=z_{1}^{N_{1}-1} z_{2}^{N_{2}-2} \tag{2.45}
\end{equation*}
$$

and integration domain $\left\{0 \leq z \leq 1 ; \hat{y}_{\min } \leq y \leq \hat{y}_{\max }\right\}$ is converted into $\left\{0 \leq z_{1,2} \leq 1\right\}$. Putting all together, we prove the identity

$$
\begin{align*}
C\left(N, M ; \alpha_{s}\right) & =C\left(\frac{N_{1}+N_{2}}{2}, \frac{N_{1}-N_{2}}{i} ; \alpha_{s}\right) \\
& =\iint_{[0,1]} d z_{1} d z_{2} z_{1}^{N_{1}-1} z_{2}^{N_{2}-1} C\left(z, \hat{y} ; \alpha_{s}\right), \tag{2.46}
\end{align*}
$$

where $(z, \hat{y})=(z, \hat{y})\left(z_{1}, z_{2}\right)$. We are just left with a double Mellin transformation of the coefficient function. We abuse a little of the notation and define

$$
\begin{equation*}
C\left(N_{1}, N_{2} ; \alpha_{s}\right) \equiv \iint_{[0,1]} d z_{1} d z_{2} z_{1}^{N_{1}-1} z_{2}^{N_{2}-1} C\left(z, \hat{y} ; \alpha_{s}\right), \tag{2.47}
\end{equation*}
$$

but Eq. (2.46) should clarify what we are doing.
We have proved a property of the Mellin-Fourier factorization of the cross section, but we have not said yet if this is useful for resummation. This is indeed the main result of Ref.[4] where it is found that resummation can be actually performed in $\left(N_{1}, N_{2}\right)$ space, retaining full rapidity dependence. The resummed cross section turns out to depend only of a single variable, $N_{1} \cdot N_{2}$. Defining

$$
\begin{equation*}
\omega=\alpha_{s} \ln \left(N_{1} N_{2}\right), \tag{2.48}
\end{equation*}
$$

it can be written as

$$
\begin{equation*}
C_{r e s}\left(\omega ; \alpha_{s}\right)=g_{0}\left(\alpha_{s}\right) \exp \mathcal{S}\left(\omega, \alpha_{s}\right) \tag{2.49}
\end{equation*}
$$

The Sudakov like exponent can be organized in the customary $N^{k}$ LL fashion

$$
\begin{equation*}
\mathcal{S}\left(\omega, \alpha_{f}\right)=\frac{1}{\alpha_{s}} g_{1}(\omega)+g_{2}(\omega)+\alpha_{s} g_{3}(\omega)+\cdots \tag{2.50}
\end{equation*}
$$

with the $g$ function defined as power series in $\omega$.
The result has the same form of that given in Eqs.(2.11-2.14), but this one contain the rapidity information, which will enter explicitly in the direct space result after the double Mellin inverse transform has been performed.

### 2.4.1 Rapidity distribution in Mellin-Mellin space

It is far from trivial the fact that the only dependence is on the single variable $\omega$. In this subsection, we verify this prediction. ${ }^{4}$ Following the notation of

[^13]ref. [2], we write
\[

$$
\begin{equation*}
\frac{d^{2} \sigma}{d Q d Y}=\frac{4 \pi \alpha^{2}}{9 Q^{3}} \sum_{i j} \int_{x_{1}^{0}}^{1} d x_{1} \int_{x_{2}^{0}}^{1} d x_{2} f_{i}\left(x_{1}\right) f_{j}\left(x_{2}\right) \frac{d \sigma_{i j}(z, u)}{d Y}, \tag{2.51}
\end{equation*}
$$

\]

where $z=\frac{Q^{2}}{s x_{1} x_{2}}$ and $u=\left(x_{1} / x_{2}\right) e^{-2 Y}$. We define the variable

$$
\begin{equation*}
y^{\prime}=\frac{u-z}{(1-z)(1+u)} \tag{2.52}
\end{equation*}
$$

and exploiting the relations in Eq. (2.44), one finds the following relations

$$
\begin{align*}
& z=z_{1} z_{2} \quad y^{\prime}=\frac{z_{2}\left(1-z_{1}^{2}\right)}{\left(1-z_{1} z_{2}\right)\left(z_{1}+z_{2}\right)},  \tag{2.53}\\
& z_{1}=\sqrt{\frac{z(1-y(1-z))}{z+y(1-z)}} \quad z_{2}=z_{1}\left[y^{\prime} \leftrightarrow 1-y^{\prime}\right] . \tag{2.54}
\end{align*}
$$

Moreover, since $x_{i}=x_{i}^{0} / z_{i}$, the following change of variables holds

$$
\begin{equation*}
d x_{1} d x_{2}=\frac{\tau}{z_{1}^{2} z_{2}^{2}} d z_{1} d z_{2} \tag{2.55}
\end{equation*}
$$

The partonic cross section can be decomposed as follows

$$
\begin{equation*}
\frac{d \sigma_{i j}}{d Y}=\frac{1}{1-z_{1} z_{2}}\left[\eta_{i j}^{(0)}+\left(\frac{\alpha_{s}}{\pi}\right) \eta_{i j}^{(1)}+\mathcal{O}\left(\alpha_{s}^{2}\right)\right] \tag{2.56}
\end{equation*}
$$

where the first two coefficients are, [2]:

$$
\begin{align*}
\eta_{i j}^{(0)} & =Q_{q}^{2}\left[\delta_{i q} \delta_{\bar{q} i}+[i \leftrightarrow j]\right] \delta(1-z)\left[\delta\left(y^{\prime}\right)+\delta\left(1-y^{\prime}\right)\right],  \tag{2.57}\\
\eta_{q \bar{q}}^{(1)} & =Q_{q}^{2} \frac{8}{3} \frac{z^{2}}{1+z}\left\{[ \delta ( y ^ { \prime } ) + \delta ( 1 - y ^ { \prime } ) ] \left[\delta(1-z)\left(2 \zeta_{2}-4\right)\right.\right. \\
& \left.+4\left[\frac{\log (1-z)}{1-z}\right]_{+}-2(1+z) \log (1-z)-\frac{1+z^{2}}{1-z} \log z+1-z\right] \\
& \left.+\left(1+\frac{(1-z)^{2}}{z} y^{\prime}\left(1-y^{\prime}\right)\right)\left[\frac{1+z^{2}}{[1-z]_{+}}\left(\frac{1}{y_{+}^{\prime}}+\frac{1}{\left[1-y^{\prime}\right]_{+}}\right)-2(1-z)\right]\right\} . \tag{2.58}
\end{align*}
$$

At LO the delta functions forces us to to pick the $q \bar{q}$ channel. At NLO also other channels contribute. We gave the corrections to the $q \bar{q}$ channel. Following $[4,3]$, we are interested in computing the large- $N_{i}$ limit of

$$
\begin{equation*}
\Delta_{i j}\left(N_{1}, N_{2}\right)=\left[\prod_{i=1,2} \int_{0}^{1} d z_{i} z_{i}^{N_{i}-1}\right] \frac{1}{z_{1} z_{2}} \frac{d \sigma_{i j}(z, u)}{d Y} . \tag{2.59}
\end{equation*}
$$

Thanks to delta functions, the LO is easily treated

$$
\begin{aligned}
& \delta(1-z) \delta\left(y^{\prime}\right)=\delta\left(1-z_{1} z_{2}\right) \delta\left(\frac{z_{2}\left(1-z_{1}^{2}\right)}{\left(1-z_{1} z_{2}\right)\left(z_{1}+z_{2}\right)}\right) \\
& \quad=\delta\left(1-z_{1} z_{2}\right) \delta\left(1-z_{1}\right) \frac{\left(1-z_{2}\right)\left(1+z_{2}\right)}{2 z_{2}}=\delta\left(1-z_{2}\right) \delta\left(1-z_{1}\right)\left(1-z_{2}\right),
\end{aligned}
$$

The case with $y^{\prime} \leftrightarrow 1-y^{\prime}$ is just the same with $z_{1} \leftrightarrow z_{2}$. Observe that we cannot evaluate the last factor giving a null term because of the overall factor in Eq. (2.56): the two cancel against each other. The full LO is

$$
\begin{equation*}
\eta_{q \bar{q}}^{(0)}=2 Q_{q}^{2} \delta\left(1-z_{1}\right) \delta\left(1-z_{2}\right) \tag{2.60}
\end{equation*}
$$

Under double Mellin transform, we get

$$
\begin{equation*}
\int_{0}^{1} d z_{1} d z_{2} z_{1}^{N_{1}} z_{2}^{N_{2}} \eta_{q \bar{q}}=2 Q_{q}^{2} \tag{2.61}
\end{equation*}
$$

N -dependent contribution can only be found starting from the next order. Looking at Eq. (2.58), the first addend is just made up of delta function, so it can be treated as the LO and it gives other constant contributions. Terms that give N -dependent contributions in Mellin space are

$$
\begin{align*}
& \frac{8}{3} Q_{q}^{2} \frac{z}{1-z^{2}} 4\left[\delta\left(y^{\prime}\right)+\delta\left(1-y^{\prime}\right)\right]\left[\frac{\log (1-z)}{1-z}\right]_{+}  \tag{2.62}\\
& \frac{8}{3} Q_{q}^{2} \frac{z\left(1+z^{2}\right)}{1-z^{2}}\left(1+\frac{(1-z)^{2}}{z} y^{\prime}\left(1-y^{\prime}\right)\right) \frac{1}{[1-z]_{+}}\left(\frac{1}{[y]_{+}}+\frac{1}{[1-y]_{+}}\right) . \tag{2.63}
\end{align*}
$$

All the other terms in Eq. (2.58) give power suppressed contribution for large $N_{i}$. Te Jacobian of transformation Eq. (2.53) is

$$
\begin{equation*}
\frac{d z d y^{\prime}}{d z_{1} d z_{2}}=\frac{2[1-y(1-z)][1-(1-y)(1-z)]}{1-z^{2}} \tag{2.64}
\end{equation*}
$$

using which we can recast the term in Eq. (2.62) as

$$
\begin{align*}
& \frac{32}{3(1-z)} \frac{z}{1+z}\left[\delta\left(y^{\prime}\right)+\delta\left(1-y^{\prime}\right)\right]\left[\frac{\log (1-z)}{1-z}\right]_{+} z_{1}^{N_{1}} z_{2}^{N_{2}} d z_{1} d z_{2} \\
& =\frac{32 z z_{1}^{N_{1}} z_{2}^{N_{2}}}{3\left(1-z^{2}\right)}\left[\delta\left(y^{\prime}\right)+\delta\left(1-y^{\prime}\right)\right]\left[\frac{\log (1-z)}{1-z}\right]_{+} \frac{\left(1-z^{2}\right) d z d y^{\prime}}{2\left[1-y^{\prime}(1-z)\right]\left[1-\left(1-y^{\prime}\right)(1-z)\right]} \\
& =\frac{16 z z_{1}^{N_{1}} z_{2}^{N_{2}}}{3}\left[\delta\left(y^{\prime}\right)+\delta\left(1-y^{\prime}\right)\right]\left[\frac{\log (1-z)}{1-z}\right]_{+} \frac{d z d y^{\prime}}{\left[1-y^{\prime}(1-z)\right]\left[1-\left(1-y^{\prime}\right)(1-z)\right]} \tag{2.65}
\end{align*}
$$

Exploiting the delta functions we can perform one integration, getting

$$
\begin{equation*}
\frac{16}{3}\left[\left[\frac{\log (1-z)}{1-z}\right]_{+} z^{N_{2}-1}+\left[\frac{\log (1-z)}{1-z}\right]_{+} z^{N_{1}-1}\right] \tag{2.66}
\end{equation*}
$$

where we have also used $z_{1}(y=0)=z_{2}\left(y^{\prime}=1\right)=1$ and $z_{2}\left(y^{\prime}=0\right)=z_{1}\left(y^{\prime}=\right.$ $1)=z$, see Eq (2.53). Using integrals Eq. (A.27) from Appendix A. 3 of this thesis, we can compute the final Mellin transform, which is, for large- $N_{i}$

$$
\begin{equation*}
\frac{8}{3}\left[\left(\log ^{2}\left(N_{1}\right)+2 \gamma_{E} \log \left(N_{1}\right)+\gamma_{E}^{2}+\zeta_{2}\right)+\left(N_{1} \rightarrow N_{2}\right)\right]+\mathcal{O}\left(\frac{1}{N_{i}}\right) . \tag{2.67}
\end{equation*}
$$

The other relevant term is

$$
\begin{equation*}
\frac{8}{3} \frac{z^{2}}{1+z}\left(1+\frac{(1-z)^{2}}{z} y(1-y)\right) \frac{1+z^{2}}{[1-z]_{+}}\left(\frac{1}{y_{+}}+\frac{1}{[1-y]_{+}}\right) \frac{1}{z(1-z)} \tag{2.68}
\end{equation*}
$$

Using the distributional identities of table S 2 in Appendix B , [20], the last expression becomes

$$
\begin{align*}
& \frac{8}{3}\left[\left(-\left(\frac{\ln \left(1-z_{1}\right)}{1-z_{1}}\right)_{+} \delta\left(1-z_{2}\right)-\left(z_{1} \leftrightarrow z_{2}\right)\right)\right. \\
& \left.+\left(\frac{1}{1-z_{1}}\right)_{+}\left(\frac{1}{1-z_{2}}\right)_{+}+\frac{\pi^{2}}{6} \delta\left(1-z_{1}\right) \delta\left(1-z_{2}\right)\right]+\cdots \tag{2.69}
\end{align*}
$$

where dots stand for contributions that would be power suppressed in the large- $N_{i}$ limit. Again, using integrals Eq. (A.27), expression (2.69) becomes

$$
\begin{align*}
-\frac{8}{3} & {\left[\frac{1}{2}\left(\log ^{2}\left(N_{1}\right)+2 \gamma_{E} \log \left(N_{1}\right)+\gamma_{E}^{2}+\frac{\pi^{2}}{6}\right)+\frac{1}{2}\left(N_{1} \leftrightarrow N_{2}\right)\right.}  \tag{2.70}\\
& \left.-\left(\gamma_{E}+\log \left(N_{1}\right)\right)\left(\gamma_{E}+\log \left(N_{2}\right)\right)-\frac{\pi^{2}}{6}\right]+\mathcal{O}\left(\frac{1}{N_{i}}\right)
\end{align*}
$$

The sum of results (2.68) and (2.70) is, at logarithmic level

$$
\begin{equation*}
2 C_{F} \log ^{2}(\bar{\omega})+\mathcal{O}\left(\frac{1}{N_{i}}\right) \tag{2.71}
\end{equation*}
$$

where we are using the notation of [3], with $\bar{\omega}=N_{1} N_{2} e^{2 \gamma_{E}}$, and introduced the group constant $C_{F}=\frac{4}{3}$. At logarithmic level, our result agrees with the one from ref. [3] ${ }^{5}$.

We have explicitly verified the announced prediction. What we are going to do in the following Chapters is consider the even less inclusive case of a double differential distribution, that is differential both in transverse momentum and rapidity of the final state. We aim at extending the double Mellin approach given here to that case.

[^14]
## Chapter 3

## The fully differential distribution

In this chapter we consider the fully differential distribution for production of a colorless final state $S$ plus other undetected radiation, $X$. For the rest of the thesis we will focus on Higgs production $(S=H)$, computed at NLO. First, we briefly explain how Higgs production process are computed in the high $m_{t}$ limit. Then, we give the general kinematics for this process. Since we are interested in soft large logarithms, understanding the kinematics is fundamental to identify the threshold limit. After that, we show how the cross section factorizes under a Mellin-Fourier transform. Finally, due to the fact that we want to explicitly compute the Mellin-Fourier transform of the partonic cross section, we perform a change of variables in order to express the partonic cross section on integration variables. The change of variable will be performed taking into account the fact that we are interested in the threshold limit.

### 3.1 Higgs production in gluon fusion: the effective interaction

In proton-proton colliders, the Higgs Boson is mainly produced through the gluon fusion channel. Gluons do not interact directly with the Higgs, so the intermediate step is a quark loop, Fig. 3.1 on the left. This results in very difficult calculations, due to the presence of loops already at LO. As a consequence, the calculation is usually performed using an Effective Field Theory (EFT) approach, in which the top quark mass is considered large compared to the Higgs mass, $m_{t o p} \gg m_{H}$. This is the same strategy which


Figure 3.1: Higgs Production in gluon fusion in the Standard Model (left) and in EFT (right).
reduces the electroweak theory to the Fermi theory for the beta decay.
By removing the top quark from the theory, it remains an effective Higgs gluon coupling, [16],

$$
\begin{equation*}
\mathcal{L}_{e f f}=-\frac{1}{4}\left(1-C_{W} H\right) G_{\mu \nu}^{a} G^{a, \mu \nu} \tag{3.1}
\end{equation*}
$$

where $C_{W}$ is called the Wilson coefficient and it is given, [24], as a series in $\alpha_{s}$,

$$
\begin{equation*}
C_{W}=\frac{\alpha_{s}}{3 \pi v}\left(1+\frac{\alpha_{s}}{4 \pi} \Delta+\mathcal{O}\left(\alpha_{s}^{2}\right)\right) \tag{3.2}
\end{equation*}
$$

with coefficients

$$
\begin{equation*}
\Delta=5 N_{c}-3 C_{F}=11, \quad N_{c}=3 \quad C_{F}=4 / 3 . \tag{3.3}
\end{equation*}
$$

The effective three fields vertex $g g H$ is depicted in Fig. 3.1 on the right. Vertices with more gluons are characterized by more power of $\alpha_{s}$. The partonic cross section that we are going to give has been computed by means of this EFT scheme.

### 3.2 Kinematics and Notations

### 3.2.1 Notations

We are going to consider the process

$$
\begin{equation*}
h_{1}\left(P_{1}\right)+h_{2}\left(P_{2}\right) \longrightarrow H(p)+X(Q), \tag{3.4}
\end{equation*}
$$

where $h_{i}$ are the colliding hadrons, $H$ is the outgoing Higgs boson and $X$ labels any extra undetected radiation.
Since we are going to consider only the gluon fusion channel, we are also able to represent the partonic process

$$
\begin{equation*}
g\left(p_{1}\right)+g\left(p_{2}\right) \longrightarrow H(p)+X(Q) \tag{3.5}
\end{equation*}
$$

where $p_{i}=x_{i} P_{i}$ are the partonic momenta, carrying a fraction $x_{i}$ of the hadron momenta. The four momenta characterizing the process are, in the hadronic frame of reference,

$$
\left.\begin{array}{rlr}
p_{1} & =\frac{\sqrt{s}}{2} x_{1}(1,0,0,1) & p_{2}=\frac{\sqrt{s}}{2} x_{2}(1,0,0,-1)  \tag{3.6}\\
p & =\left(E_{H}, \vec{p}_{T}, p_{L}\right) & Q
\end{array}\right)\left(Q_{0}, \vec{Q}_{T}, Q_{L}\right), ~ \$
$$

where $s=\left(P_{1}+P_{2}\right)^{2}$ is the squared centre of mass hadronic energy. By defining the rapidity and the transverse mass of the Higgs as

$$
y=\frac{1}{2} \ln \frac{E_{H}+p_{L}}{E_{H}-p_{L}}, \quad m_{T}^{2}=m_{H}^{2}+p_{T}^{2}
$$

and after a little algebra, one finds that the Higgs momentum may be rewritten as

$$
\begin{equation*}
p=\left(m_{T} \cosh (y), \vec{p}_{T}, m_{T} \sinh (y)\right) \tag{3.7}
\end{equation*}
$$

We recall the definitions given in Sec.2.3:

$$
\xi_{p}=\frac{p_{T}^{2}}{M_{H}^{2}} \quad a\left(\xi_{p}\right)=\left(\sqrt{\xi_{p}+1}+\sqrt{\xi_{p}}\right)^{2}
$$

so that

$$
\begin{equation*}
m_{T}=m_{H} \sqrt{1+\xi_{p}} \tag{3.8}
\end{equation*}
$$

Just as matter of convenience, we are going to parameterize the transverse momentum by means of the dimensionless variables $\xi_{p}$ and $a\left(\xi_{p}\right)$, instead of $p_{T}$. Note that the limit in which the Higgs becomes collinear, $p_{T} \rightarrow 0$, is represented by either $\xi_{p} \rightarrow 0$ or $a \rightarrow 1$. In general, from the definition and the non-negativity of $\xi_{p}$, follows that $1 \leq a\left(\xi_{p}\right)$.
Keeping in mind the total momentum conservation law

$$
\begin{equation*}
p_{1}+p_{2}=p+Q, \tag{3.9}
\end{equation*}
$$

we can define the customary Mandelstam invariants

$$
\begin{align*}
& \hat{s}=\left(p_{1}+p_{2}\right)^{2}=(p+Q)^{2}=s x_{1} x_{2}, \\
& \hat{t}=\left(p_{1}-Q\right)^{2}=\left(p_{2}-p\right)^{2}=m_{H}^{2}-\sqrt{s} x_{2} m_{T} e^{y},  \tag{3.10}\\
& \hat{u}=\left(p_{2}-Q\right)^{2}=\left(p_{1}-p\right)^{2}=m_{H}^{2}-\sqrt{s} x_{1} m_{T} e^{-y} .
\end{align*}
$$

## CHAPTER 3. THE FULLY DIFFERENTIAL DISTRIBUTION

These invariants satisfy ${ }^{1}$

$$
\begin{equation*}
Q^{2}+m_{H}^{2}=\hat{s}+\hat{t}+\hat{u} \tag{3.11}
\end{equation*}
$$

It is more convenient to compute the kinematic boundaries in the partonic frame of reference. We know how to shift the rapidity from the hadronic to the partonic frame, Eq. (2.29). In this frame, the four momenta in Eq. (3.6) become
$p_{1}=\frac{\sqrt{\hat{s}}}{2}(1,0,0,1)$
$p=\left(m_{T} \cosh \hat{y}, \vec{p}_{T}, m_{T} \cosh \hat{y}\right)$
$Q=\left(\sqrt{\hat{s}}-m_{T} \cosh (\hat{y}),-\vec{p}_{T},-m_{T} \sinh \hat{y}\right)$,
where we made use of Eq. (3.9). A few, maybe obvious, remarks about the Lorentz boost from the hadronic to the partonic frame:

- $\vec{p}_{T}$ it is left unchanged. The boost is performed along the third axis;
- From the previous observation and from the rewriting of Eq. (3.7) it is clear the Higgs momentum just changes in its rapidity variable. This is why this parametrization proves manageable.

In the partonic frame we also rewrite $\hat{t}, \hat{u}$ as

$$
\begin{equation*}
\hat{t}=m_{H}^{2}-\sqrt{\hat{s}} m_{T} e^{\hat{y}} \quad \hat{u}=m_{H}^{2}-\sqrt{\hat{s}} m_{T} e^{-\hat{y}} . \tag{3.13}
\end{equation*}
$$

In order to make the calculation simple, we just need a little more of notation. We define, as we did in the previous Chapter, the scaling variables

$$
\begin{align*}
& \tau:=\frac{m_{H}^{2}}{s} \quad z:=\frac{\tau}{x_{1} x_{2}}=\frac{m_{H}^{2}}{s}  \tag{3.14}\\
& \tau^{\prime}:=\tau a\left(\xi_{p}\right) z^{\prime}:=z a\left(\xi_{p}\right) .
\end{align*}
$$

### 3.2.2 Kinematic

The two condition which defines the physical region are

$$
\begin{align*}
Q_{0} & \geq 0  \tag{3.15a}\\
Q^{2} & \geq 0 . \tag{3.15b}
\end{align*}
$$

[^15]By requiring the condition Eq. (3.15b) on $Q$, Eq. (3.12), one finds the following condition for $\hat{y}$

$$
\begin{equation*}
\cosh (\hat{y}) \leq \frac{a+z^{\prime}}{\sqrt{z^{\prime}}(a+1)} \tag{3.16}
\end{equation*}
$$

Noting that the hyperbolic cosine is always $\geq 1$, it follows that

$$
z^{\prime} \leq 1 \quad \vee \quad z^{\prime} \geq a^{2}
$$

The second solution does not satisfy the positive energy condition Eq. (3.15a). From this result, the inversion of Eq. (3.16) and the fact that $x_{1,2}$ have upper limit 1 by definition, it follows that

$$
\left\{\begin{array}{l}
\tau^{\prime} \leq z^{\prime} \leq 1  \tag{3.17}\\
\hat{y}_{\min } \leq \hat{y} \leq \hat{y}_{\max }
\end{array}\right.
$$

where the rapidity extrema are

$$
\begin{align*}
\hat{y}_{\max } & =-\hat{y}_{\min }=\ln \left[\frac{a+z^{\prime}+\sqrt{\left(a^{2}-z^{\prime}\right)\left(1-z^{\prime}\right)}}{\sqrt{z^{\prime}}(a+1)}\right]  \tag{3.18}\\
& =\ln \left[\frac{a+z^{\prime}+R}{\sqrt{z^{\prime}}(a+1)}\right]
\end{align*}
$$

with the further definition

$$
\begin{equation*}
R:=\sqrt{\left(a^{2}-z^{\prime}\right)\left(1-z^{\prime}\right)} . \tag{3.19}
\end{equation*}
$$

Boosting back the solution for the rapidity one should find the lower extrema for the partonic fraction $x_{1}$ and $x_{2}$, that is the double differential analogous condition of Eq. (2.30). This is not algebraically trivial, because it involves the solution of highly non linear inequalities. On the other hand, as we are going to show below, we do not really care about those limit for what we are going to do.
The important result to keep in mind are the extrema in expression (3.17). We expect the cross section to show enhanced contribution on those boundaries. In the following sections we will consider the explicit expression for the cross section, verifying that this is indeed the case.

In the previous chapter, we promised we would have said how to recover the single differential kinematic. It is sufficient to look at the explicit
expression of conditions Eqs.(3.15a,3.15b) and vary $p_{T}$ (for the rapidity distribution) or $y$ (for the transverse momentum distribution) to find the widest range possible for the $x_{i}$ to satisfy those constraint. Not surprisingly, they correspond to taking $p_{T}=0$ and $y=0$, respectively.

### 3.3 Fully differential distribution at NLO and its Mellin-Fourier transform

The fully differential cross section for the production of a Higgs boson of transverse momentum $p_{T}$ and rapidity $y$ can be computed in pQCD and written as

$$
\begin{equation*}
\frac{d \sigma}{d \xi_{p} d y}=\sum_{i, j} \int_{0}^{1} d x_{1} d x_{2} f_{i / h_{1}}\left(x_{1}, \mu_{F}^{2}\right) f_{j / h_{2}}\left(x_{2}, \mu_{F}^{2}\right) \frac{d \hat{\sigma}_{i j}}{d \xi_{p} d y}, \tag{3.20}
\end{equation*}
$$

where the partonic subprocess is $i j \longrightarrow H+X$ and $i, j=g, q_{f}, \bar{q}_{f}$, for every flavour $f$.

The partonic subprocess is computed perturbatively in the strong coupling, $\alpha_{s}\left(\mu_{R}\right)$,

$$
\begin{equation*}
\frac{d \hat{\sigma}_{i j}}{d \xi_{p} d y}=\frac{\sigma_{0} a\left(\xi_{p}\right)}{z^{\prime}}\left[\frac{\alpha_{s}\left(\mu_{R}\right)}{2 \pi} G_{i j}^{(1)}+\left(\frac{\alpha_{s}\left(\mu_{R}\right)}{2 \pi}\right)^{2} G_{i j}^{(2)}+\cdots\right] \tag{3.21}
\end{equation*}
$$

where $\sigma_{0}$ is the tree level inclusive cross section

$$
\begin{equation*}
\sigma_{0}=\frac{\pi}{64}\left(\frac{\alpha_{s}\left(\mu_{R}\right)}{3 \pi v}\right)^{2} . \tag{3.22}
\end{equation*}
$$

Kinematic boundaries in Eq. (3.17) allow us to write

$$
\begin{align*}
\frac{d \sigma}{d \xi_{p} d y}\left(\tau^{\prime}, y, \xi_{p}\right) & =\int_{0}^{1} d x_{1} d x_{2} d z^{\prime} \int_{\hat{y}_{\text {min }}}^{\hat{y}_{\text {max }}} d \hat{y} \delta\left(\tau^{\prime}-x_{1} x_{2} z^{\prime}\right)  \tag{3.23}\\
& \times \delta\left(\hat{y}-y+\frac{1}{2} \ln \frac{x_{1}}{x_{2}}\right) \frac{d \hat{\sigma}}{d \xi_{p} d y}\left(z^{\prime}, \hat{y}, \xi_{p}\right)
\end{align*}
$$

with implicit dependence on the coupling and on the factorization and renormalization scales.
As we did for the case of rapidity distributions, Eq. (2.36), we take a combined Mellin-Fourier transform, with respect to the scaling variable $\tau^{\prime}$ and
the rapidity of the Higgs boson, $y$, and we get

$$
\begin{align*}
\frac{d \sigma}{d \xi_{p} d y}\left(N, M, \xi_{p}\right) & \equiv \int_{0}^{1} d \tau^{\prime} \tau^{\prime N-1} \int_{y_{\min }}^{y_{\max }} d y e^{i M y} \frac{d \sigma}{d \xi_{p} d y}\left(\tau^{\prime}, y, \xi_{p}\right) \\
& =f_{1}(N+i M / 2) f_{2}(N-i M / 2) \frac{d \hat{\sigma}}{d \xi_{p} d y}\left(N, M, \xi_{p}\right) \tag{3.24}
\end{align*}
$$

where

$$
\begin{align*}
f_{1,2}(N \pm i M / 2) & =\int_{0}^{1} d x x^{N \pm i M / 2-1} f_{1,2}(x),  \tag{3.25}\\
C\left(N, M, \xi_{p}\right) \equiv \frac{d \hat{\sigma}}{d \xi_{p} d y}(N, M) & =\int_{0}^{1} d z^{\prime} z^{N-1} \int_{\hat{y}_{\text {min }}}^{\hat{y}_{\text {max }}} d \hat{y} e^{i M \hat{y}} \frac{d \hat{\sigma}}{d \xi_{p} d y}\left(z^{\prime}, \hat{y}, \xi_{p}\right) . \tag{3.26}
\end{align*}
$$

The $C$ coefficient, or rather its Mellin-Fourier transform in Eq. (3.26), is the object of interest of this thesis, in particular its threshold limit. Exploiting the symmetry $\hat{y} \leftrightarrow-\hat{y}$, it is more convenient to compute $C\left(N, M, \xi_{p}\right)$ as

$$
\begin{align*}
& C\left(N, M, \xi_{p}\right)=\int_{0}^{1} d z^{\prime} \int_{0}^{\hat{y}_{\max }} d \hat{y}\left(e^{i M \hat{y}} z^{\prime N-1}+e^{-i M \hat{y}} z^{\prime N-1}\right) C\left(z^{\prime}, \hat{y}, \xi_{p}\right) \\
&=\int_{0}^{1} d z^{\prime} \int_{\frac{1}{t_{\max }}}^{1} d t {\left[\left(t^{i M-1} t_{\max }^{i M} z^{\prime N-1}+t^{-i M-1} t_{\max }^{-i M} z^{\prime N-1}\right)\right.} \\
&\left.\times C\left(z^{\prime}, \ln (t)+\ln \left(t_{\max }\right), \xi_{p}\right)\right] \tag{3.27}
\end{align*}
$$

where we have defined

$$
\begin{align*}
& t:=e^{\hat{y}} / e^{\hat{y}_{\text {max }}},  \tag{3.28}\\
& t_{\text {max }}:=e^{\hat{y}_{\text {max }}} . \tag{3.29}
\end{align*}
$$

We now turn to the perturbative expression for the partonic process in the gluon fusion channel, that is $i=j=g$.

### 3.3.1 The LO

following the notation of ref. [16], at LO we have

$$
\begin{equation*}
G_{i j}^{(1)}=g_{i j} \delta\left(Q^{2}\right), \tag{3.30}
\end{equation*}
$$

where, for $i=j=g$

$$
\begin{equation*}
g_{g g}=N_{c}\left(\frac{m_{H}^{8}+\hat{s}^{4}+\hat{t}^{4}+\hat{u}^{4}}{\hat{s} \hat{t} \hat{u}}\right) . \tag{3.31}
\end{equation*}
$$

Eq. (3.30) tells us that at LO there is just one parton recoiling against the Higgs boson, so $Q$ is actually the four momentum of a real single particle, and it must satisfy the on-shellness condition.

### 3.3.2 $\mathcal{O}\left(\alpha_{s}\right)$ corrections

The order $\mathcal{O}\left(\alpha_{s}\right)$ corrections come from two different sources: the interference between one loop diagrams and the Born level and one parton real emission diagrams.
The final result can be decomposed into a singular function plus a regular one

$$
\begin{equation*}
G_{i j}^{(2)}=G_{i j}^{(2 s)}+G_{i j}^{(2 R, n s)}, \tag{3.32}
\end{equation*}
$$

where "singular" means that it contains all the enhanced logarithms as $p_{T} \rightarrow$ 0 and all the plus distributions arising from the cancellation of soft and collinear singularities.
We are interested in the singular part since it gives enhanced threshold contributions. Following again the notation of [16], we have

$$
\begin{align*}
G_{g g}^{(2 s)} & =\delta\left(Q^{2}\right)\left\{\left(\Delta+\delta+N_{c} U\right) g_{g g}\right. \\
& \left.+\left(N_{c}-N_{f}\right) \frac{N_{c}}{3}\left[\left(m_{H}^{4} / \hat{s}\right)+\left(m_{H}^{4} / \hat{t}\right)+\left(m_{H}^{4} / \hat{u}\right)+m_{H}^{2}\right]\right\} \\
& +\left\{\left(\frac{1}{-\hat{t}}\right)\left[-P_{g g}\left(z_{t}\right) \ln \frac{\mu_{F}^{2} z_{t}}{(-\hat{t})}+p_{g g}\left(z_{t}\right)\left(\frac{\ln \left(1-z_{t}\right)}{1-z_{t}}\right)_{+}\right]_{g g g, t}\left(z_{t}\right)\right.  \tag{I}\\
& +\left(\frac{1}{-\hat{t}}\right)\left[-2 n_{f} P_{q g}\left(z_{t}\right) \ln \frac{\mu_{F}^{2}}{Q^{2}}+2 n_{f} C_{q g}^{\epsilon}\left(z_{t}\right)\right] g_{q g, t}\left(z_{t}\right)  \tag{II}\\
& +\left(\frac{z_{t}}{-\hat{t}}\right)\left(\left(\frac{\ln \left(1-z_{t}\right)}{1-z_{t}}\right)_{+}-\frac{\ln Q_{T}^{2} z_{t} /(-\hat{t})}{\left(1-z_{t}\right)_{+}}\right)  \tag{III}\\
& \times \frac{N_{c}^{2}}{2}\left[\frac{\left(m_{H}^{8}+\hat{s}^{4}++\hat{t}^{4}+\hat{u}^{4}+Q^{8}\right)+z_{t} z_{u}\left(m_{H}^{8}+\hat{s}^{4}++\left(\hat{t} / z_{t}\right)^{4}+\left(\hat{u} / z_{u}\right)^{4}+Q^{8}\right)}{\hat{s} \hat{t} \hat{u}}\right] \\
& -\left(\frac{z_{t}}{-\hat{t}}\right)\left(\frac{1}{1-z_{t}}\right)_{+} \frac{\beta_{0}}{2} N_{c}\left(\frac{\left.m_{H}^{8}+\hat{s}^{4}++z_{t} z_{u}\left(\left(\hat{t} / z_{t}\right)^{4}+\left(\hat{u} / z_{u}\right)^{4}\right)\right)}{\hat{s} \hat{t} \hat{u}}\right)  \tag{IV}\\
& +[(t, \hat{t}) \leftrightarrow(u, \hat{u})]\} \\
& N_{c}^{2}\left[\frac{\left.\left(m_{H}^{8}+\hat{s}^{4}+Q^{8}+\left(\hat{u} / z_{u}\right)^{4}\right)+\left(\hat{t} / z_{t}\right)^{4}\right)\left(Q^{2}+Q_{T}^{2}\right)}{\hat{s}^{2} Q^{2} Q_{T}^{2}}+\right. \\
& \frac{2 m_{H}^{4}\left(\left(m_{H}^{2}-\hat{t}\right)^{4}+\left(m_{H}^{2}-\hat{u}\right)^{4}+\hat{u}^{4}+\hat{t}^{4}\right)}{\hat{s} \hat{t} \hat{u}\left(m_{H}^{2}-\hat{t}\right)\left(m_{H}^{2}-\hat{u}\right)} \frac{1}{p_{T}^{2}} \ln \frac{p_{T}^{2}}{Q_{T}^{2}}, \tag{3.33}
\end{align*}
$$

where the following variables have been defined

$$
\begin{gather*}
z_{t, u}=\frac{-\hat{t}, \hat{u}}{Q^{2}-\hat{t}, \hat{u}},  \tag{3.34}\\
Q_{T}^{2}=Q^{2}+p_{T}^{2} \tag{3.35}
\end{gather*}
$$

The function $g_{g g}$ is the same of that at LO , whereas $g_{g g, t / u}$ can be found in [16].

The AP splitting function are, [16]

$$
\begin{align*}
& P_{g g}(z)=N_{c}\left[\frac{1+z^{4}+(1-z)^{4}}{z(1-z)_{+}}\right]+\beta_{0} \delta(1-z),  \tag{3.36}\\
& P_{q g}(z)=C_{F}\left[\frac{1+(1-z)^{2}}{z}\right] .
\end{align*}
$$

Plus distribution are defined in Appendix [A.2]. Finally, the numerator of the splitting function is given by

$$
\begin{equation*}
p_{g g}(z)=(1-z) P_{g g}(z)=N_{c}\left[\frac{1+z^{4}+(1-z)^{4}}{z}\right] . \tag{3.37}
\end{equation*}
$$

All the function that we have not explicitly written, namely $\Delta, \delta, U$, give regular contribution near threshold. Their analytical expression can be found in Ref.[16].

### 3.4 Fully differential distribution at NLO: threshold behaviour

We are now ready to use our kinematical analysis to show where enhanced soft contributions come from. Plus distributions arise in cancellation of soft singularities. As a consequence, the regularized endpoint, namely $z_{t, u}=1$, should correspond to the kinematical extrema we found, (3.17):

- if $z^{\prime} \rightarrow 1$, then the partonic process is in threshold and all the radiation become soft;
- if $|\hat{y}| \rightarrow \hat{y}_{\text {max }}$ then radiation on one of the two incoming partons become soft;

Clearly, the first case includes the second, because whenever $z^{\prime} \rightarrow 1$, then $\hat{y}_{\text {max }}=-\hat{y}_{\text {min }} \rightarrow 0$ and $\hat{y}$ is constrained accordingly.

In order to verify this, we rewrite $z_{t, u}$ in terms of $z^{\prime}, \hat{y}$ and the fixed parameter $a\left(\xi_{p}\right)$. We have

$$
\begin{equation*}
z_{t}=\sqrt{z^{\prime}} \frac{(a+1) e^{\hat{y}}-2 \sqrt{z^{\prime}}}{2 a-\sqrt{z^{\prime}}(a+1) e^{-\hat{y}}} \quad z_{u}=z_{t}[\hat{y} \leftrightarrow-\hat{y}] . \tag{3.38}
\end{equation*}
$$

Then, the claim is verified just substituting the boundaries in Eq. (3.17) into Eq. (3.38).

A careful analysis of each terms in Eq. (3.33) shows that there are not other singular contributions.

We know where possible soft enhancement come from, so we can proceed with the study of the threshold behaviour of the partonic cross section.

### 3.4.1 Threshold evaluation of regular contribution

Regular contribution are function $f=f\left(z^{\prime}, t ; \xi_{p}\right)$ that can be evaluated for $z^{\prime}=t=1$, that is for $Q^{2}=0$ with correction of order $\mathcal{O}(1-t)$ or $\mathcal{O}(1-z)$ which are of no interest in the search for soft enhanced contribution.

Starting with the LO, the $g_{g g}$ of Eq. (3.31) is given by

$$
\begin{equation*}
\left.g_{g g}\right|_{z^{\prime}=t=1}=N_{c} m_{H}^{2}\left[\frac{8+8 a^{4}+(a-1)^{4}}{2 a(a-1)}\right] . \tag{3.39}
\end{equation*}
$$

At NLO we need to consider the following (see (I-II) in Eq. (3.33))

$$
\begin{equation*}
\left.g_{g g, t \hat{u} u}\right|_{z^{\prime}=t=1}=N_{c} m_{H}^{2}\left[\frac{1+a^{4}+\frac{1}{16}(a-1)^{4}+\frac{16 a^{4} \xi_{p}^{4}}{\left(a-1^{4}\right)}}{\xi_{p} a^{2}}\right] . \tag{3.40}
\end{equation*}
$$

Moreover, regular coefficient of lines (III-IV) can be evaluated at threshold. It is not particularly interesting to give them here. They just reduce to contributions proportional to $\left.g_{g g}\right|_{z^{\prime}=t=1}$.

As final observation, we want point out that the expression in Eq. (3.40) becomes equal to that of Eq. (3.39) if the small- $p_{T}$ limit is taken. In fact, they both behaves as

$$
\begin{equation*}
\sim \frac{2 N_{c} m_{H}^{2}}{\xi_{p}} \quad \text { as } \quad \xi_{p} \rightarrow 0 . \tag{3.41}
\end{equation*}
$$

We need to proceed very carefully here, since the small $-p_{t}$ limit is not trivial. We are considering two limits, the threshold one and the small- $p_{T}$ limit. It turns out that this two limits do not commute. We postpone this issue for later. For the moment, we just keep in mind the small- $p_{T}$ limit of Eq. (3.41). In this case, the limit is taken after the threshold one.

### 3.4.2 Change of variable of singular contribution

We now turn to the change of variable in the singular contribution. We need to express the plus distribution in terms of the variables $z^{\prime}$ and $t$. The full
derivation can be found in Appendix [B.1] .
Here we give the result:

$$
\begin{align*}
\frac{z_{a}}{-a}\left(\frac{1}{1-z_{a}}\right)_{+} & =\frac{z^{\prime}}{m_{H}^{2} R}\left[\left(\frac{1}{1-t}\right)_{+}-\delta(1-t) \ln (\rho)\right. \\
& \left.+\frac{1}{t-1+\frac{R}{\omega}}+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right]  \tag{3.42}\\
\frac{z_{a}}{-a}\left(\frac{\ln \left(1-z_{a}\right)}{1-z_{a}}\right)_{+} & =\frac{z^{\prime}}{m_{H}^{2} R}\left[\left(\frac{\ln 1-t}{1-t}\right)_{+} \frac{1}{1-\frac{\omega}{R}(1-t)}\right. \\
& -\left(\frac{1}{1-t}\right)_{+} \frac{\ln (\rho)}{1-\frac{\omega}{R}(1-t)}+\frac{1}{2} \ln ^{2}(\rho) \delta(1-t)  \tag{3.43}\\
& \left.+\left(\frac{1}{1-t}\right)_{+} \frac{\ln \left(1-\frac{\omega}{R}(1-t)\right)}{1-\frac{\omega}{R}(1-t)}+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right]
\end{align*}
$$

where $R$ is defined in Eq. (3.19) and has threshold behaviour given by

$$
\begin{equation*}
R=\sqrt{\left(a^{2}-1\right)\left(1-z^{\prime}\right)}+\mathcal{O}\left((\sqrt{1-z})^{2}\right) \tag{3.44}
\end{equation*}
$$

We have also defined the following

$$
\begin{align*}
& \rho \equiv \frac{1}{2} \sqrt{\frac{a-1}{a+1}} \frac{1}{\sqrt{1-z^{\prime}}}  \tag{3.45}\\
& \omega \equiv \frac{a+1}{2} . \tag{3.46}
\end{align*}
$$

At LO the main contribution is given by $\delta\left(Q^{2}\right)$ :

$$
\begin{equation*}
\delta\left(Q^{2}\right)=\frac{z^{\prime}}{m_{H}^{2} R}\left[\delta(1-t)+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right] \tag{3.47}
\end{equation*}
$$

A validity check One possible way for verifying the validity of this change of variables is the following one. We consider the inner integral in Eq. (3.27) and set $M=0$. In that case, the integral over the $t$ variable gives nothing else than the rapidity integrated cross section, computed in the threshold
limit. As an example, consider the contribution labelled by (IV). We have

$$
\begin{align*}
(I V) & =-\left(\frac{z_{t}}{-\hat{t}}\right)\left(\frac{1}{1-z_{t}}\right)_{+} \frac{\beta_{0}}{2} N_{c}\left(\frac{\left.m_{H}^{8}+\hat{s}^{4}++z_{t} z_{u}\left(\left(\hat{t} / z_{t}\right)^{4}+\left(\hat{u} / z_{u}\right)^{4}\right)\right)}{\hat{s} \hat{t} \hat{u}}\right) \\
& =-\left.\frac{\beta_{0}}{2} g_{g g}\right|_{z^{\prime}=t=1} \frac{z^{\prime}}{m_{H}^{2} R}\left[\left(\frac{1}{1-t}\right)_{+}-\delta(1-t) \ln (\rho)\right. \\
& \left.+\frac{1}{t-1+\frac{R}{\omega}}+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right] . \tag{3.48}
\end{align*}
$$

Apart from the delta function, which is trivial, the integral over the plus distribution and the one over the rational function are given in Appendix B. 2 and at the end of Appendix B.1.2. It turns out that

$$
\begin{align*}
\int_{t_{\text {min }}}^{1} d t(I V) & =\frac{z^{\prime}}{m_{H}^{2} \sqrt{a^{2}-1} \sqrt{1-z^{\prime}}}\left\{\frac{1}{2} \ln ^{2}\left(1-z^{\prime}\right)\right. \\
& \left.+2 \ln (2) \ln \left(1-z^{\prime}\right)+2 \ln ^{2}(2)-\frac{\pi^{2}}{6}+\mathcal{O}\left(\sqrt{1-z^{\prime}}\right)\right\} \tag{3.49}
\end{align*}
$$

All other contributions may be computed in the same way using results from Appendix. B.2. all the integrals reproduce the correct threshold behaviour of the $p_{T}$-differential partonic cross section.

CHAPTER 3. THE FULLY DIFFERENTIAL DISTRIBUTION

## Chapter 4

## Fully differential threshold behaviour in Mellin-Fourier space

From the previous chapter we know where soft enhanced contribution come from. Moreover, we have also shown that under a Mellin-Fourier transform taken with respect to the variable $\left(z^{\prime}, t\right)$ the cross section factorizes into the product between the Mellin transform of the non perturbative PDFs and the Mellin-Fourier transform of the partonic cross section. We expect soft large logarithms to appear as logarithm of the Mellin-Fourier conjugated variables. Given these facts, in this chapter we are going to look into the master integral in Eq. (3.27).
First, we present the full double transform performed on the small $p_{T}$-limit of the cross section. Then, we attempt to generalize for arbitrary, fixed, transverse momentum. We compute the integration of the first order of a suitable expansion and formulate a conjecture for the all order result.

### 4.1 The small $p_{T}$ limit

The integral in Eq. (3.27) is highly non trivial, due to the presence of $t_{\max }$ which enters in two different ways:

- As a factor in the integrand, with exponent $i M$;
- As lower limit in the integration domain;

We recall here the expression for $t_{\max }$ :

$$
\begin{equation*}
t_{\max }^{-1}\left(z^{\prime}, \xi_{p}\right)=\frac{\sqrt{z^{\prime}}(a+1)}{a+z^{\prime}+R}, \tag{4.1}
\end{equation*}
$$

where

$$
\begin{equation*}
R=\sqrt{\left(a^{2}-z^{\prime}\right)\left(1-z^{\prime}\right)} . \tag{4.2}
\end{equation*}
$$

Even when the inner integral of Eq. (3.27) is performed on a delta function $\delta(1-t)$, then we also need to compute

$$
\begin{equation*}
\int_{0}^{1} d z^{\prime} z^{\prime N-1} \frac{1}{\sqrt{1-z}}\left(t_{\max }^{i M}+t_{\max }^{-i M}\right) \tag{4.3}
\end{equation*}
$$

an integral which, given the expressions Eqs.(4.1, 4.2), we have not been able to compute exactly. The situation is worse when plus distribution are involved.

We decide to consider the more manageable situation of the small $p_{T}$ limit. The reason is that as $p_{T} \rightarrow 0$, the kinematic approaches that of the single differential rapidity distribution, a situation we we know how to deal with. Moreover, we know that the result near the threshold will be made up of contributions that are either independent from $p_{T}$ or proportional to $\log \left(p_{T}\right)$.
We are going to deal with a double limit:

- The threshold limit, represented by $N \rightarrow \infty$ and, possibly, $i M \rightarrow \pm \infty$;
- The small $p_{T}$ limit;

It is really important to keep in mind that this two limit cannot be freely interchanged. Consider, for example, Eq. (4.1), and its threshold expansion at fixed $a\left(\xi_{p}\right)$

$$
\begin{align*}
t_{\max } & =1-\sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}+\mathcal{O}\left(\left(\sqrt{1-z^{\prime}}\right)^{2}\right)  \tag{4.4}\\
& =1-\frac{\sqrt{1-z^{\prime}}}{2} \sqrt{a-1}+\mathcal{O}\left((\sqrt{a-1})^{2}\right)+\mathcal{O}\left(\left(\sqrt{1-z^{\prime}}\right)^{2}\right)
\end{align*}
$$

On the other hand, if we took the small $p_{T}$ limit first

$$
\begin{align*}
t_{\max } & =\sqrt{z^{\prime}}+\mathcal{O}\left((a-1)^{2}\right) \\
& =1-\frac{1}{2}\left(1-z^{\prime}\right)+\mathcal{O}\left(\left(1-z^{\prime}\right)^{2}\right)+\mathcal{O}\left((a-1)^{2}\right) \tag{4.5}
\end{align*}
$$

The kinematical origin of these differences can be easily understood. The emission of one parton of transverse momentum parameter $\xi_{p}$, in a process characterized by the partonic scaling variable $z$, has phase space proportional to

$$
\begin{equation*}
d \Phi_{k} \propto \frac{d z d k_{T}^{2}}{\sqrt{(1-z)^{2}-4 z \xi_{p}}} \tag{4.6}
\end{equation*}
$$

One must choose one order or the other in the evaluation of the threshold and small $p_{T}$ limit. Until now, we have considered the more intricate situation in which we consider the threshold limit at fixed $p_{T}$. The hope is that, after the threshold expansion, the small $-p_{T}$ limit could simplify the final calculation. For the moment, we postpone the problem and step back for a while: in the next subsection we treat the case in which the first limit we take is the small- $p_{T}$ one. This calculation should reproduce the already known result of transverse momentum resummation.

### 4.1.1 Threshold evaluation of small- $p_{T}$ distribution

As we have just said, the small- $p_{T}$ limit may display new divergences. We cannot rely on the change of variables Section 3.4.2 anymore, and neither on the results of Section 3.4.1. For example, consider Eq. (3.31) and, in particular, its denominator. At fixed $p_{T}$, the Mandelstam invariants admits the threshold evaluation

$$
\begin{align*}
& \left.\hat{t}\right|_{\left[\hat{y}=\hat{y}_{\text {max }}, z^{\prime}=1\right]}=\frac{m_{H}^{2}}{2 z^{\prime}}\left(z^{\prime}-a-R\right)_{z^{\prime}=1}=m_{H}^{2}(1-a)  \tag{4.7}\\
& \left.\hat{u}\right|_{\left.\hat{y}=\hat{y}_{\text {max }}, z^{\prime}=1\right]}=\frac{m_{H}^{2}}{2 z^{\prime}}\left(z^{\prime}-a+R\right)_{z^{\prime}=1}=m_{H}^{2}(1-a) .
\end{align*}
$$

Consider $\hat{t}, \hat{u}$ in terms of the $z_{1,2}$ variables of Eq. (2.44) and take the small $p_{T}$ limit

$$
\begin{align*}
& \lim _{a \rightarrow 1}-\hat{t}=\lim _{a \rightarrow 1} \frac{m_{H}^{2}}{z_{2}}\left(\frac{a+1}{2}-z_{2}\right)=\frac{m_{H}^{2}}{z_{2}}\left(1-z_{2}\right) \\
& \lim _{a \rightarrow 1}-\hat{u}=\lim _{a \rightarrow 1} \frac{m_{H}^{2}}{z_{1}}\left(\frac{a+1}{2}-z_{1}\right)=\frac{m_{H}^{2}}{z_{1}}\left(1-z_{1}\right) . \tag{4.8}
\end{align*}
$$

## CHAPTER 4. FULLY DIFFERENTIAL THRESHOLD BEHAVIOUR IN MELLIN-FOURIER SPACE

Since the kinematic in this limit is the same as that of the rapidity distribution, the threshold limit is encoded in $z_{1,2} \rightarrow 1$. Whenever Mandelstam invariants appear in the denominator, we cannot take the small momentum limit so easily as we did in the last equation. We need to single out the divergence in term of a vanishing $p_{T}$.

The correct change of variables for this particular limit can be found in Appendix [B.3]. Here we show with an example how it is possible to isolate the most divergent - in power of $p_{T}$ - term.
Consider the following

$$
\begin{equation*}
\frac{1}{-\hat{t}} \propto \frac{1}{\left(\frac{a+1}{2}-z_{2}\right)}, \tag{4.9}
\end{equation*}
$$

and a test function $f$. Under integration

$$
\begin{align*}
& \int_{0}^{1} d z f(z) \frac{1}{\frac{a+1}{2}-z}=\int_{0}^{1} d z \frac{f(z)-f(1)}{\frac{a+1}{2}-z}+f(1) \int_{0}^{1} d z \frac{1}{\frac{a+1}{2}-z}  \tag{4.10}\\
& =\int_{0}^{1} d z f(z)\left[\left(\frac{1}{1-z}\right)_{+}+\delta(1-z) \ln \left(\frac{a+1}{a-1}\right)\right]+\mathcal{O}(a-1) .
\end{align*}
$$

The neglected contributions come entirely from the expansion of the denominator of the first term.
In other words, the following identity between distributions holds

$$
\begin{equation*}
\frac{1}{\left(\frac{a+1}{2}-z\right)}=\left(\frac{1}{1-z}\right)_{+}+\delta(1-z) \ln \left(\frac{a+1}{a-1}\right)+\mathcal{O}(a-1) . \tag{4.11}
\end{equation*}
$$

Here the most divergent term is not divergent at all, it is of order $\left(p_{T}\right)^{0}$ times a contribution which is $p_{T}$ finite or proportional to a logarithmic divergence in $p_{T}$. We show now, with another example, how a power divergence in $p_{T}$ could arise. We start squaring Eq. (4.9), and proceed in the same way as above.

$$
\begin{align*}
& \int_{0}^{1} d z f(z) \frac{1}{\left(\frac{a+1}{2}-z\right)^{2}}=\int_{0}^{1} d z \frac{f(z)-f(1)}{\left(\frac{a+1}{2}-z\right)^{2}}+f(1) \int_{0}^{1} d z \frac{1}{\left(\frac{a+1}{2}-z\right)^{2}} \\
& =\int_{0}^{1} d z g_{a}(z) \frac{1}{\left(\frac{a+1}{2}-z\right)}+\frac{4}{a^{2}-1} \int_{0}^{1} d z f(z) \delta(1-z) \tag{4.12}
\end{align*}
$$

where we have defined the regular function

$$
\begin{equation*}
g_{a}(z):=\frac{f(z)-f(1)}{\frac{a+1}{2}-z} . \tag{4.13}
\end{equation*}
$$

Note that, also in the limit $a \rightarrow 1, g_{1}(z)$ is regular ${ }^{1}$. We can exploit Eq. (4.11) and write

$$
\begin{align*}
& \int_{0}^{1} d z f(z) \frac{1}{\left(\frac{a+1}{2}-z\right)^{2}}=\int_{0}^{1} d z\left[g_{1}(z)\left[\left(\frac{1}{1-z}\right)_{+}+\delta(1-z) \ln \left(\frac{a+1}{a-1}\right)\right]\right. \\
& \left.+\frac{4}{a^{2}-1} f(z) \delta(1-z)\right]+\left(p_{T}-\text { vanishing terms }\right) \\
& =\int_{0}^{1} d z f(z) \frac{4}{a^{2}-1}\left[\delta(1-z)+\left(p_{T}-\text { vanishing terms }\right)\right] . \tag{4.14}
\end{align*}
$$

Consequently, retaining only the most power divergent contribution, the result is a delta function. Following analogous procedures, it is possible to treat each line in Eq. (3.33). Before we give our results, we need to say something about what happens to the coefficient $g_{g g}$ and $g_{g g, t / u}$. The $g_{g g} \mathrm{~s}$ can be treated as follows. First, we define

$$
\begin{equation*}
\tilde{g}_{g g}=\frac{\hat{t} \hat{u}}{m_{H}^{4}} g_{g g}, \tag{4.15}
\end{equation*}
$$

which can be evaluated at threshold and gives

$$
\begin{equation*}
\left.\tilde{g}_{g g}\right|_{a=1, z_{1}=z_{2}=1}=2 m_{H}^{2} N_{c} . \tag{4.16}
\end{equation*}
$$

Similarly

$$
\begin{equation*}
\left.g_{g g, \hat{t}}\right|_{\text {thres. }}=\left.g_{g g, \hat{u}}\right|_{t h r e s .}=\left.\frac{1}{\xi_{p} m_{H}^{4}} \tilde{g}_{g g}\right|_{\text {thres. }} . \tag{4.17}
\end{equation*}
$$

Importantly, the $g_{g g, \hat{t} / \hat{u}}$ functions show an extra enhancement in $p_{T}$, as Eq. (4.17) shows. When computing the change of variable using the strategy outlined above, this fact has important consequences in counting powers of $p_{T}$ in the denominator. The outcome is that only lines in Eq. (3.33) with AP splitting functions leaves singular (in threshold) contribution also in the small- $p_{T}$ limit. All other pieces give at most delta function. We can convince ourselves that this is correct. In fact, in the limit of small- $p_{T}$, we know that all N -dependent logarithmic contributions come from anomalous dimensions: in this case, the AP splitting functions.

[^16]Coming to the kinematics, we have already anticipated that it trivially reduces to one of the differential rapidity distribution one and, when the change of variable in Eq. (2.44) is performed, it leads to a double Mellin transform.

Using results in Appendix. A.3.1, we find that the double Mellin transform of the small- $p_{T}$ limit of Eq. (3.33), is given by

$$
\begin{align*}
& \int_{0}^{1} d z_{1} z_{1}^{N_{1}-1} \int_{0}^{1} d z_{2} z_{2}^{N_{2}-1}\left[G_{g g}^{(2 s)}\right]= \\
& \frac{1}{\xi_{p}} \frac{\tilde{g}_{g g}}{m_{H}^{2}}\left\{\operatorname { l n } ( \frac { \mu _ { F } ^ { 2 } } { m _ { H } ^ { 2 } } ) \left[-4 N_{c}\left(\gamma_{E}^{2}+\gamma_{E} \ln \left(N_{1} N_{2}\right)+\ln \left(N_{1}\right) \ln (N 2)\right)\right.\right. \\
& \left.+\left(2 \gamma_{E}+\ln \left(N_{1} N_{2}\right)\right)\left(\beta_{0}-N_{c} \ln \left(\xi_{p}\right)\right)+\beta_{0} \ln \left(\xi_{p}\right)\right]+2 N_{c}\left[-3 \gamma_{E} \ln ^{2}\left(N_{1} N_{2}\right)\right. \\
& -2 \gamma_{E}\left(\gamma_{E}^{2}+\frac{\pi^{2}}{6}\right)+\left(\gamma_{E}^{2}+\frac{\pi^{2}}{6}\right) \ln \left(N_{1} N_{2}\right)-\ln \left(N_{1}\right) \ln \left(N_{2}\right)\left(\gamma_{E}+\ln \left(N_{1} N_{2}\right)\right) \\
& \left.-\frac{1}{8} \ln ^{2}\left(\xi_{p}\right)\left(2 \gamma_{E}+\ln \left(N_{1} N_{2}\right)\right)-\frac{1}{2} \ln \left(\xi_{p}\right)\left(\frac{1}{2} \ln ^{2}\left(N_{1} N_{2}\right)+\gamma_{E} \ln \left(N_{1} N_{2}\right)\right)\right] \\
& +\beta_{0}\left[-\frac{1}{4} \ln ^{2}\left(\xi_{p}\right)+\frac{1}{2} \ln ^{2}\left(N_{1} N_{2}\right)+\gamma_{E} \ln \left(N_{1} N_{2}\right)+\left(\frac{\pi^{2}}{6}+\gamma_{E}^{2}\right)\right. \\
& \left.\left.-\ln \left(N_{1}\right) \ln \left(N_{2}\right)\right]+\mathcal{O}\left(\xi_{p}\right)+\mathcal{O}\left(\frac{1}{N_{i}}\right)\right\} \tag{4.18}
\end{align*}
$$

This result is compatible with the one given in Ref.[7]. In that paper, a Fourier transform with respect to the transverse momentum and a double Mellin transform with respect to $z_{1,2}$, are taken. Reciprocal space variables are $b$ and $\left(N_{1}, N_{2}\right)$, respectively. Taking the limit $b \rightarrow \infty$, only the contributions from $p_{T} \rightarrow 0$ are picked up. In this limit, the cross section is shown to exponentiate as

$$
\begin{equation*}
C^{\text {res }} \propto \mathcal{H}\left(N_{1}, N_{2} ; \alpha_{s}\right)\left\{\exp \left[\mathcal{G}\left(N_{1}, b ; \alpha_{s}\right)\right] \exp \left[\mathcal{G}\left(N_{2}, b ; \alpha_{s}\right)\right]\right\} . \tag{4.19}
\end{equation*}
$$

The two exponential carry the full $p_{T}(b)$ dependence. Since they can be easily put together in a single exponential, it follows that each contribution proportional to $b$ can depend only through the product $N_{1} N_{2}$. Our result in Eq. (4.18) agrees with this.

### 4.2 Threshold behaviour for fixed $p_{T}$

In this section we deal with the case in which we first consider the threshold limit for the cross section and then, possibly, the small $-p_{T}$ limit. We encounter a difficulty, which can be easily understood if we look at the integrand in Eq.(4.3). Since we would like to take the $\pm \infty$ limit of the variable $i M$, we cannot take the small $-p_{T}$ limit, not even on the threshold expanded expression Eq.(4.4), because of that divergent exponent. It is not a priori clear how to take this limit before the explicit evaluation of all the transforms and of the large ( $N, i M$ ) asymptotic expansion. This is exactly what we would have preferred to avoid in the first place.

We propose another way of proceeding by "brute force". Coming back to the general expression for the integral, eq.(3.27), we perform the following Taylor expansions for $z^{\prime} \rightarrow 1$ :

- $t_{\text {max }}$ appears as limit of integration. Use the expansion for $t_{\text {max }}$, first line in Eq.(4.4);
- Expand $t^{i M}$ around $t=1$. This is justified by the fact that the domain of integration is pushed toward the endpoint $t=1$ as $z \rightarrow 1$;
- Expand also $t_{\text {max }}^{i M}$.

If the first is supposed to be harmless to our calculation, the second and the third expansions are quite subtle, due to the presence of increasing powers of $i M$ besides those of $(1-z)$ or $(1-t)$. In fact, we cannot discard terms of the form $(1-z)^{k}$ if they are multiplied by the same power of $i M$. Our aim is to use integrals computed on the first - relevant - order expansion to formulate a conjecture for the general result.

Define the expansion parameter

$$
\begin{equation*}
b:=i M . \tag{4.20}
\end{equation*}
$$

It should not be surprising that this calculation turns out to be analytically long and tedious, due to the numerous series expansions. Accordingly, we are not going to give the full computation. We just outline the procedure and give the results. General integrals may be found in Appendix [B.4], leaving out all the algebra.
We start making the following observation: the result is completely symmetric under the exchange $b \leftrightarrow-b$, reflecting the forward-backward symmetry

## CHAPTER 4. FULLY DIFFERENTIAL THRESHOLD BEHAVIOUR IN MELLIN-FOURIER SPACE

in rapidity distribution. Consequently, overall expansions need to be performed at least at order $\mathcal{O}\left(b^{2}\right)$, because the first order will always cancel. The order $\mathcal{O}\left(b^{0}\right)$ corresponds to set $b=0$ and it reproduces the rapidity integrated results, as we have already pointed out at the end of Section 3.4.2. The following expansions is everything we need

$$
\begin{align*}
\frac{1}{t_{\max }} & =1-b \sqrt{\frac{a-1}{a+1}} \sqrt{1-z}+\frac{b^{2}}{2}\left(\frac{a-1}{a+1}\right)(1-z) \\
& +\mathcal{O}\left(b^{3}\right)+\mathcal{O}\left((\sqrt{1-z})^{3 / 2}\right),  \tag{4.21}\\
t^{b} & =1-b(1-t)+\frac{b^{2}}{2}(1-t)^{2}+\mathcal{O}\left(b^{3}\right)+\mathcal{O}\left((\sqrt{1-z})^{3 / 2}\right) . \tag{4.22}
\end{align*}
$$

At this level we need to retain also the order $\mathcal{O}(b)$, since we always encounter the product $t^{b}\left(\frac{1}{t_{\text {max }}}\right)^{i M}$ in which two contributions of order $\mathcal{O}(b)$ may give rise to a term of order $\mathcal{O}\left(b^{2}\right)$.
At a computational level, we are left with integrals that are similar to those we encountered when we computed the rapidity integrated result as a check. Differences appears as higher powers of $(1-t)$ or $(1-z)$ at the numerator. Combining Eqs. $(4.21,4.22)$ we find

$$
\begin{align*}
t_{\text {max }}^{b} t^{b}+[b \leftrightarrow-b]=2+ & 2 b^{2}\left(-\sqrt{\frac{a-1}{a+1}} \sqrt{1-z}(1-t)\right.  \tag{4.23}\\
& \left.+\frac{1}{2}\left(\frac{a-1}{a+1}\right)(1-z)+\frac{1}{2}(1-t)^{2}\right)+\mathcal{O}\left(b^{4}\right) .
\end{align*}
$$

We just need to substitute this last expansion into Eq.(3.27) and do the calculation. The result can be decomposed as follows

$$
\begin{align*}
& \int_{0}^{1} d z^{\prime} z^{\prime N-1} \int_{\frac{1}{t_{\text {max }}}}^{1} d t\left[t_{\max }^{b} t^{b}+[b \leftrightarrow-b]\right] G_{g g}^{(2 s)} \\
& =\frac{g_{g g} \mid z^{\prime}=t=1}{\sqrt{a^{2}-1}} \frac{\sqrt{\pi}}{\sqrt{N}} \times\left\{C_{0}\left(N, \xi_{p}\right)+\frac{b^{2} \sqrt{\frac{a-1}{a+1}}}{4 N^{2}} C_{2}\left(N, \xi_{p}\right)+\mathcal{O}\left(b^{4}\right)\right\}, \tag{4.24}
\end{align*}
$$

where $C_{0}$ is the result of the rapidity integrated cross section, whereas $C_{2}$ is the order $\mathcal{O}\left(b^{2}\right)$ correction.

We have:

$$
\begin{align*}
C_{0} & =\beta_{0}\left(2 \gamma_{E}+2 \ln (N)\right)+3 N_{c}\left(\gamma_{E}^{2}-3 \ln ^{2}(2)+\ln ^{2}(N)+2 \gamma_{E} \ln (N)+\frac{\pi^{2}}{6}\right) \\
& +2 N_{c} \ln \left(\frac{2 \xi_{p}}{a-1}\right)\left(2 \gamma_{E}+2 \ln (N)\right)-2 \ln \left(\frac{2 \mu_{F}^{2}}{m_{H}^{2}(a-1)}\right)\left(\beta_{0}-4 N_{c} \gamma_{E}-4 \ln (N)\right),  \tag{4.25}\\
C_{2} & =\beta_{0}\left(2 \gamma_{E}+2 \ln (N)\right)+3 N_{c}\left[56 \ln ^{2}(2)+\frac{103}{24} \ln (2)+\frac{25}{9}-\frac{10}{3} \pi^{2}\right. \\
& \left.+\ln \left(\frac{a-1}{a+1}\right)\left(17 \ln (2)+\frac{19}{3}\right)+\left(17 \ln (2)+\frac{19}{3}\right)\left(2-\gamma_{E}-2 \ln (2)-\ln (N)\right)\right] \\
& +2 N_{c} \ln \left(\frac{2 \xi_{p}}{a-1}\right)\left(2 \gamma_{E}+2 \ln (N)\right)-2 \ln \left(\frac{2 \mu_{F}^{2}}{m_{H}^{2}(a-1)}\right)\left(\beta_{0}-4 N_{c} \gamma_{E}-4 N_{c} \ln (N)\right) . \tag{4.26}
\end{align*}
$$

The complete NLO Mellin-Fourier transform would be given by the whole series in $b$. Though we have not been able to sum up the whole series, still we can give the general result as a conjecture and verify that our result agree with the appropriate expansion of the conjectured result. This is left for the last section.

### 4.3 A conjecture for the resummed cross section

We can no more rely on Eq.(4.19) because it is valid only for $p_{T} \rightarrow 0$. From a diagrammatic point of view, soft radiation has different sources:

- emissions from external legs of incoming partons;
- emissions from the recoiling parton;

As a consequence, we could expect a resummed structure of the form

$$
\begin{equation*}
C^{\text {res }}\left(N_{1}, N_{2} ; \alpha_{s}\right) \propto \exp \left(\mathcal{S}\left(N_{1} N_{2}, \alpha_{s}\right)\right) \times\left[\exp \left(\mathcal{G}\left(N_{1} ; \alpha_{s}\right)\right)+\exp \left(\mathcal{G}\left(N_{2} ; \alpha_{s}\right)\right)\right], \tag{4.27}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{S}\left(N_{1} N_{2} ; \alpha_{s}\right) & =\ln \left(N_{1} N_{2}\right) \cdot f\left(\ln \left(N_{1} N_{2}\right) ; \alpha_{s}\right),  \tag{4.28}\\
f\left(\ln \left(N_{1} N_{2} ; \alpha_{s}\right)\right) & =\sum_{n \geq 1} f_{n}\left(\alpha_{s} \ln \left(N_{1} N_{2}\right)\right) \alpha_{s}^{n},  \tag{4.29}\\
\mathcal{G}\left(N_{i} ; \alpha_{s}\right) & =\ln \left(N_{i}\right) \cdot g\left(\ln \left(N_{i}\right) ; \alpha_{s}\right),  \tag{4.30}\\
g\left(\ln \left(N_{i}\right) ; \alpha_{s}\right) & =\sum_{n \geq 1} g_{n}\left(\alpha_{s} \ln \left(N_{i}\right)\right) \alpha_{s}^{n} . \tag{4.31}
\end{align*}
$$

Each $f_{n}$ and $g_{n}$ can be written as a series expansion in their argument

$$
\begin{align*}
& f_{n}(\lambda)=\sum_{k \geq 0} f_{n k} \alpha_{s}^{k},  \tag{4.32}\\
& g_{n}(\lambda)=\sum_{k \geq 0} g_{n k} \alpha_{s}^{k} . \tag{4.33}
\end{align*}
$$

We need to say how we can map this structure into something that depends logarithmically on $N$ and on $b^{2}$ in the first order of a series expansion. We note that

$$
\begin{equation*}
N_{1} \cdot N_{2}=\left(N+\frac{b}{2}\right)\left(N-\frac{b}{2}\right)=N^{2}\left(1-\frac{b^{2}}{4 N^{2}}\right) \tag{4.34}
\end{equation*}
$$

so that

$$
\begin{equation*}
\ln \left(N_{1} N_{2}\right)=\ln \left(N^{2}\right)+\ln \left(1-\frac{b^{2}}{4 N^{2}}\right)=2 \ln (N)-\frac{b^{2}}{4 N^{2}}+\cdots \tag{4.35}
\end{equation*}
$$

Expanding Eq.(4.27) we get

$$
\begin{align*}
C^{\text {res }} \propto & 1+\alpha_{s}\left\{\left[\ln ^{2}(N)\left(4 f_{11}+2 g_{11}\right)+2 \ln (N)\left(f_{20}+g_{20}\right)\right]\right.  \tag{4.36}\\
& \left.+\frac{M^{2}}{4 N^{2}}\left[\left(4 f_{11}-2 g_{11}\right) \ln (N)+f_{20}+g_{20}-2 g_{11}\right]\right\}+\cdots
\end{align*}
$$

Consider the following contribution, that are nothing else that the building blocks of Eq. (4.24)

$$
\begin{align*}
A= & 3 N_{C} ;  \tag{4.37}\\
B= & 2 \beta_{0}+6 N_{c} \gamma_{E}+4 N_{c} \ln \left(\frac{2 \xi_{p}}{a-1}\right)+8 N_{c} \ln \left(\frac{2 \mu_{F}^{2}}{m_{H}^{2}(a-1)}\right) ;  \tag{4.38}\\
C= & \sqrt{\frac{a-1}{a+1}}\left(2 \beta_{0}-6 N_{c}\left(17 \ln (2)+\frac{19}{3}\right)+4 N_{c} \ln \left(\frac{2 \xi_{p}}{a-1}\right)\right. \\
& \left.-8 \ln \left(\frac{2 \mu_{F}^{2}}{M_{H}^{2}(a-1)}\right)\right) ;  \tag{4.39}\\
D= & \sqrt{\frac{a-1}{a+1}}\left[2 \beta_{0} \gamma_{E}+6 N_{c}\left(28 \ln ^{2}(2)+\frac{103}{48} \ln (2)+\frac{25}{18}-\frac{5}{3} \pi^{2}\right.\right.  \tag{4.40}\\
& +\frac{1}{2} \ln \left(\sqrt{\frac{a-1}{a+1}}\right)\left(17 \ln (2)+\frac{19}{3}\right)+\left(17 \ln (2)+\frac{19}{3}\right)(1 \\
& \left.\left.\left.-\frac{\gamma_{E}}{2}-\ln (2)\right)\right)\right]+4 N_{c} \ln \left(\frac{2 \xi_{p}}{a-1}\right)-2 \ln \left(\frac{2 \mu_{F}^{2}}{m_{H}^{2}(a-1)}\right)\left(\beta_{0}-4 N_{c} \gamma_{E}\right) . \tag{4.41}
\end{align*}
$$

and then we require that

$$
\begin{array}{cc}
f_{11}=\frac{A+C}{8} & g_{11}=\frac{A-C}{4}  \tag{4.42}\\
f_{20}=B-D-\frac{A-C}{2} & g_{20}=A-C+2 D-B .
\end{array}
$$

The last assignments ensure that our result agrees with this conjecture Eq.(4.27), with proportionality constant given by

$$
\frac{\left.g_{g g}\right|_{z^{\prime}=t=1}}{\sqrt{a^{2}-1}} \frac{\sqrt{\pi}}{\sqrt{N}}
$$

In conclusion, the conjecture can be written as

$$
\begin{align*}
C^{\text {res }} & =\left(\left.g_{g g}\right|_{z^{\prime}=t=1} \frac{\sqrt{\pi}}{\sqrt{N}}\right) \exp \left(\mathcal{S}\left(N_{1} N_{2}, \alpha_{s}\right)\right)  \tag{4.43}\\
& \times\left[\exp \left(\mathcal{G}\left(N_{1} ; \alpha_{s}\right)\right)+\exp \left(\mathcal{G}\left(N_{2} ; \alpha_{s}\right)\right)\right]
\end{align*}
$$

CHAPTER 4. FULLY DIFFERENTIAL THRESHOLD BEHAVIOUR IN MELLIN-FOURIER SPACE

## Chapter 5

## Conclusion

The object of study of this thesis were soft large logarithms and their resummation. Soft resummation prescription are know for inclusive and single differential distribution. Our purpose was the understanding of the general structure of soft large logarithms in the case of fully differential cross sections. Soft large logarithms arise in kinematic regions where the centre of mass energy is just enough to produce the final state, that is when extra radiations become soft.

For inclusive cross section, through the variable $z=m_{H}^{2} / \hat{s}$, the threshold region is identified by $z \rightarrow 1$. The cross section is written as a multiplicative convolution and then factorized under a Mellin transform. Following resummation prescriptions, the large- N limit is taken and soft large logarithms in direct space are mapped into logarithms of the conjugated variable N .

The transition to $p_{T}$ distribution requires a rescaling of the relevant soft variable $z$ by a $p_{T}$-dependent coefficient: $z^{\prime}=z \cdot a\left(p_{T}\right)$. After the rescaling everything else is quite alike the inclusive case. A Mellin transform is taken and large logarithms of the conjugated variable $N$ are searched.
More difficult is the case of rapidity distribution. The single variable $z$ is no more sufficient to describe the soft region. Also the partonic rapidity $\hat{y}$ need to be considered. The right recipe turns out to be a Mellin-Fourier (M-F) transform. The M-F transform can be mapped into a double Mellin M-M transform by a change of variables. Large logarithm are functions of either one of two Mellin variables: $\left(N_{1}, N_{2}\right)$. Finally, not trivially, resummation theory for rapidity distribution ensures that the dependence on this two variables is always given by their product $\omega=N_{1} N_{2}$.

In this thesis we investigated the transition from single differential to double differential distributions. We expected that a combination of the two recipes, that is the $z$ rescaling and the double transform would be the right one. And it was indeed the case: after the rescaling from $z$ to $z^{\prime}$, the cross section still factorizes under M-F. Nevertheless, for a fixed value of both rapidity and transverse momentum, kinematics complicates drastically and so do the M-F transform. In particular, we are no more able to get a M-M transform by means of a simple change of variables.

We considered Higgs production, computed at NLO, producing different results: we have explicitly verified the non trivial prediction of a single variable dependence for rapidity distribution. For fully differential distribution, we have shown how to isolate terms which gives rise to threshold enhanced contributions and we have expressed them on the integration variables. Then, we considered the small $p_{T}$ limit and computed the MellinFourier transform in that case. Finally, we gave a conjecture for the result in the fully differential case. The conjecture is build on the result of the Mellin-Fourier transform performed on a suitable expansions of the integrand. In particular, we expected soft large logarithm to appear as function of the Mellin variable $N$ and the Fourier variable $b$.

Starting from this point, there are mainly two purposes that can be pursued. One possibility is to further investigate the Mellin-Fourier integral, with the aim of obtaining the exact NLO expression in conjugate space. That would give us the exact structure of large logarithms to be resummed. The other possibility could be consider general threshold resummation for fully differential distribution, and try to formulate a general argument for a resummed prediction. The former could be of great help to the latter, which, in turns, gives the exact resummation recipe.

## Appendix A

## Plus Distributions

In this Appendix we give the definition and general properties of the Mellin transform. We also define plus distribution and consider their behaviour under a Mellin transform.

## A. 1 Mellin Transform

Whenever we have a function defined over the range $0 \leq x \leq 1$, we can consider its Mellin transform, given by

$$
\begin{equation*}
\tilde{f}(N) \equiv \mathcal{M}[f](N) \equiv \int_{0}^{1} d x x^{N-1} f(x) \tag{A.1}
\end{equation*}
$$

The inverse transform is

$$
\begin{equation*}
f(x)=\mathcal{M}^{-1}[\tilde{f}](x)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} d N x^{-N} \tilde{f}(N), \tag{A.2}
\end{equation*}
$$

where $c$ must be greater of rightmost singularity, which exist due to the presence of a convergence abscissa.

The Mellin transform is strictly related to the Laplace transform, as it can be seen by the replacing $x=e^{-t}$.

## A.1.1 Convolution and Mellin transform

Throughout this thesis we have often come across with integral convolution of the form

$$
\begin{align*}
& (f \otimes g)(x)=\int_{x}^{1} \frac{d y}{y} f(y) g\left(\frac{x}{y}\right) \\
& \quad \int_{0}^{1} d y \int_{0}^{1} d z f(y) g(z) \delta(x-y z) \tag{A.3}
\end{align*}
$$

From the second line of Eq. (A.3), it is clear that it is commutative

$$
\begin{equation*}
f \otimes g=g \otimes f \tag{A.4}
\end{equation*}
$$

and that it can be generalized to an arbitrary number of function

$$
\begin{equation*}
\left(f_{1} \otimes \cdots \otimes f_{n}\right)(x):=\int_{0}^{1} d y_{1} \cdots \int_{0}^{1} d y_{n} f_{1}\left(y_{1}\right) \cdots f_{n}\left(y_{n}\right) \delta\left(x-y_{1} \cdots y_{n}\right) \tag{A.5}
\end{equation*}
$$

Multiplicative convolution factorizes into an algebraic product under the Mellin transform

$$
\begin{align*}
\mathcal{M}[f \otimes g](N) & =\int_{0}^{1} d x x^{N-1} \int_{0}^{1} d y \int_{0}^{1} d z f(y) g(z) \delta(x-y z)  \tag{A.6}\\
& =\left(\int_{0}^{1} d y y^{N-1} f(y)\right)\left(\int_{0}^{1} d z z^{N-1} g(y)\right)=\tilde{f}(N) \tilde{g}(N)
\end{align*}
$$

## A. 2 The Plus distribution

Given a function $f$, we define the plus distribution of $f$ with respect to $z=1$ the point, as the distribution $[f(z)]_{+}$that acts, when integrated with a test function $g$, through the prescription

$$
\begin{equation*}
\int_{0}^{1} d z[f(z)]_{+} g(z) \equiv \int_{0}^{1}(g(z)-g(1)) f(z) \tag{A.7}
\end{equation*}
$$

We are interested in plus distributions because they usually arise in the cancellation of soft divergences. Observe that

$$
\begin{equation*}
\int_{0}^{1} d z[f(z)]_{+}=0 \tag{A.8}
\end{equation*}
$$

Consider a function $f$ which behaves, as $z \rightarrow 1$, as

$$
\begin{equation*}
f(z) \sim(1-z)^{-\alpha}, \quad \alpha<2 \tag{A.9}
\end{equation*}
$$

and it is regular elsewhere, or at least in the range $[0,1)$. In general, the integral with a regular test function $g$ does not exist

$$
\begin{equation*}
\int_{0}^{1} f(z) g(z)=\infty \tag{A.10}
\end{equation*}
$$

But, a degree of divergence such as the one of Eq. (A.9) is properly regularized by the plus prescription. In fact, in the region $z \sim 1$, we have

$$
\begin{align*}
{[g(z)-g(1)] f(z) } & \sim(1-z)^{-\alpha}(1-z) \frac{d g}{d z}(z=1)+\cdots  \tag{A.11}\\
& \sim(1-z)^{1-\alpha}+\cdots
\end{align*}
$$

whose integral is convergent, if $\alpha<2$. This means that the following class of function is properly regularized

$$
\begin{equation*}
\frac{\log ^{k} 1-z}{1-z} . \tag{A.12}
\end{equation*}
$$

This gives us the class of functions for which a plus prescription with endpoint 1 is well defined: it is the set of all function which diverges at most as indicated in Eq. (A.9). Obviously, a different plus prescription, that is a plus prescription with respect to some other end point, can be defined in
the same way.
If we consider a regular function, then

$$
\begin{equation*}
[f(z)]_{+}=f(z)-\delta(1-z) \int_{0}^{1} d z f(z) \tag{A.13}
\end{equation*}
$$

otherwise, if $f$ is not regular at the end point $z=1$, then the following representation is more suitable

$$
\begin{equation*}
[f(z)]_{+}=\lim _{\epsilon \rightarrow 0^{+}}\left[\theta(1-z-\epsilon) f(z)-\delta(1-z) \int_{0}^{1-\epsilon} d z f(z)\right] \tag{A.14}
\end{equation*}
$$

where the limit is understood to be taken after integration with a test function.

Actually, Eq. (A.14) is a better definition of a plus prescription than Eq. (A.7), because it is not restricted to integral over $[0,1]$.

## A.2.1 A useful identity

The following distributional identity holds true

$$
\begin{equation*}
(1-z)^{-1+\epsilon}=\frac{1}{\epsilon} \delta(1-z)+\left(\frac{1}{1-z}\right)_{+}+\epsilon\left(\frac{\ln 1-z}{1-z}\right)_{+}+\mathcal{O}\left(\epsilon^{2}\right) . \tag{A.15}
\end{equation*}
$$

In fact, given a test function $f$, we have

$$
\begin{align*}
& \int_{0}^{1} d z f(z)(1-z)^{-1+\epsilon}=\int_{0}^{1} d z[f(z)-f(1)](1-z)^{-1+\epsilon} \\
& +f(1) \int_{0}^{1} d z(1-z)^{-1+\epsilon} . \tag{A.16}
\end{align*}
$$

The last term can be evaluated exactly:

$$
\begin{equation*}
f(1) \int_{0}^{1} d z(1-z)^{-1+\epsilon}=f(1) \frac{1}{\epsilon}=\frac{1}{\epsilon} \int_{0}^{1} d z f(z) \delta(1-z) . \tag{A.17}
\end{equation*}
$$

The first one can be written as a series in $\epsilon$, exploiting the following expansion

$$
\begin{equation*}
(1-z)^{-1+\epsilon}=\frac{1}{1-z} e^{\epsilon \ln (1-z)}=\frac{1}{1-z}+\epsilon \frac{\ln (1-z)}{1-z}+\mathcal{O}\left(\epsilon^{2}\right) . \tag{A.18}
\end{equation*}
$$

Putting all together we prove the identity.

## A. 3 Large-N behaviour of the Mellin transform

We are interested in the large N behaviour of the Mellin transform. We are going to show now that the case of a distribution is fundamentally different from the regular function one. We start with the following theorem

Theorem 1. Let $f$ be a real function and $\mathcal{M}[f](N)$ its Mellin transform. Then $|\mathcal{M}[f](N)|$ is bounded by a decreasing function for real $N$ and tends to 0 as $N \rightarrow \infty$.

Proof. We start noting that

$$
\begin{equation*}
|\mathcal{M}[f](N)| \leq \int_{0}^{1} d z\left|z^{N-1} f(z)\right|=\mathcal{M}[\mid f](N), \tag{A.19}
\end{equation*}
$$

where we used the fact that $z$ is positive and $N$ is a real variable.
We just need to show that $\mathcal{M}[|f|](N)$ is the search bounding decreasing function.

$$
\begin{equation*}
\frac{d}{d N} \mathcal{M}[|f|](N)=\int_{0}^{1} d z \log z z^{N-1}|f(z)| \leq 0, \tag{A.20}
\end{equation*}
$$

because $\log z$ is negative in the integration range.
By noting that as $N \rightarrow \infty$ the integrand approaches 0 almost everywhere and by using the dominated convergence theorem, we get the proof.

Trivial examples of the previous theorem are given by monomial integrand. The Mellin transform of a monomial of degree $M$, behaves as $\sim \frac{1}{N+M}$ for large N .

The situation is completely different when distribution are involved. A very simple example is provided by delta function. The Mellin transform of a delta $\delta\left(z-z_{0}\right)$ is exponentially decreasing in $N$ or exactly zero if $0 \leq z_{0}<1$ but it is constant if $z_{0}=1$.
The more interesting case of a plus distribution can be characterized as follows

Theorem 2. Let $f$ be a real function which is singular in $z=1$ and let $(f)_{+}$be the plus distribution defined from it. Let $\mathcal{M}\left[(f)_{+}\right](N)$ be its Mellin transform. Then $\left|\mathcal{M}\left[(f)_{+}\right](N)\right|$ is bounded by an increasing function which diverges as $N \rightarrow \infty$.

## A.3.1 The Mellin transform of Plus distribution

We are going to compute the exact asymptotic expansion and show that the Mellin transform of plus distributions actually diverge in the limit of large $N$, with logarithmic behaviour.
we start considering the identity Eq. (A.15) and defining the generating integral

$$
\begin{equation*}
G(\epsilon):=\int_{0}^{1} d z z^{N-1}(1-z)^{-1+\epsilon}-\frac{1}{\epsilon} . \tag{A.21}
\end{equation*}
$$

From $G$, the Mellin transform of plus distribution may be computed using

$$
\begin{equation*}
\mathcal{M}\left[\left(\frac{\ln ^{k}(1-z)}{1-z}\right)_{+}\right](N)=\left.\frac{d^{k}}{d \epsilon^{k}} G(\epsilon)\right|_{\epsilon=0} . \tag{A.22}
\end{equation*}
$$

The generating function can be expressed in term fo the Euler Beta function, Eq. (C.12), as

$$
\begin{equation*}
G(\epsilon)=B(N, \epsilon)-\frac{1}{\epsilon} . \tag{A.23}
\end{equation*}
$$

The pole is necessary to cancel the analogous one in the $\epsilon \rightarrow 0$ limit of the Beta function. Expanding in power of $\epsilon$ using Eqs.(C.9,C.10), we have

$$
\begin{align*}
G(\epsilon) & =\frac{\Gamma(\epsilon) \Gamma(N)}{\Gamma(\epsilon+N)}-\frac{1}{\epsilon}=\left(\frac{1}{\epsilon}-\gamma_{E}\right.  \tag{A.24}\\
& \left.+\frac{\epsilon}{2}\left(\frac{\pi^{2}}{6}+\gamma_{E}^{2}\right)+\cdots\right) \frac{\Gamma(N)}{\Gamma(N+\epsilon)}-\frac{1}{\epsilon} .
\end{align*}
$$

Using the expansion in Eq. (C.9)

$$
\begin{equation*}
\frac{\Gamma(N)}{\Gamma(N+\epsilon)}=1-\psi_{0}(N) \epsilon+\frac{1}{2}\left(\psi_{0}^{2}(N)-\psi_{1}(N)\right) \epsilon^{2}+\cdots \tag{A.25}
\end{equation*}
$$

Putting all together, and exploiting Eq. (C.11), we find, in the large-N limit

$$
\begin{align*}
& G(\epsilon)= \\
& =-\ln (N)-\gamma_{E} \\
& +\epsilon\left(\frac{1}{2} \ln (N)^{2}+\gamma_{E} \ln (N)+\frac{1}{2}\left(\gamma_{E}^{2}+\frac{\pi^{2}}{6}\right)\right)  \tag{A.26}\\
& +\mathcal{O}\left(\epsilon^{2}\right),
\end{align*}
$$

A.3. Large-N behaviour of the Mellin transform
from which one can read the asymptotic expansion of plus distributions. We are only interested in the first two

$$
\begin{gather*}
\mathcal{M}\left[\left(\frac{1}{1-z}\right)_{+}\right](N)=-\ln (N)-\gamma_{E}+\mathcal{O}\left(\frac{1}{N}\right)  \tag{A.27}\\
\mathcal{M}\left[\left(\frac{\ln (1-z)}{1-z}\right)_{+}\right](N)=\frac{1}{2} \ln ^{2}(N)+\gamma_{E} \ln (N)+\frac{1}{2}\left(\gamma_{E}^{2}+\frac{\pi^{2}}{6}\right)+\mathcal{O}\left(\frac{1}{N}\right) . \tag{A.28}
\end{gather*}
$$

These last two integrals have been used at the end of Ch. 2. In general, at order $\epsilon^{k}$, the asymptotic gives a term proportional to $\ln ^{k-1} N$. This proves the (leading logarithmic) conversion, valid for $N \rightarrow \infty$,

$$
\begin{equation*}
\left(\frac{\ln ^{k}(1-z)}{1-z}\right)_{+} \longleftrightarrow \ln ^{k+1}(N) \tag{A.29}
\end{equation*}
$$

## Appendix B

## Changes of variables

In this appendix we compute all change of variables we have used throughout Ch. 3 and Ch. 4 of this thesis. First we perform the full derivation of Eqs. $(3.42,3.43)$. Then, we show how to perform a integral we used to make a validity check on our change of variables. Finally, we consider the limit of small- $p_{T}$ of the partonic cross section Eq. (3.21). This limit should be $p_{T}$ divergent. Here, we single out the most divergent contributions from the NLO cross section.

## B. 1 Changes of variables in plus distributions

## B.1.1 Change of variable

Eqs. $(3.42,3.43)$ are labelled by the two Mandelstam invariant, $\hat{t}, \hat{u}$. We consider the case labelled by the Mandelstam invariant $\hat{t}$. The other one can be computed analogously.

We want to express the plus distributions

$$
\begin{equation*}
\frac{z_{t}}{-\hat{t}}\left(\frac{\ln ^{k}\left(1-z_{t}\right)}{1-z_{t}}\right)_{+} \tag{B.1}
\end{equation*}
$$

as function of the variables $t$ and $z^{\prime}$, defined in Eq. (3.14,3.28). Since we are interested in the limit $t \rightarrow 1$ we can expand our expression in this limit. We define, as in Eq. (3.19),

$$
R=\sqrt{\left(a^{2}-z^{\prime}\right)\left(1-z^{\prime}\right)} .
$$

First, the prefactor

$$
\begin{equation*}
\frac{z_{t}}{-\hat{t}}=\frac{2 z^{\prime}}{m_{H}^{2}\left(a-z^{\prime}+R\right)}+\mathcal{O}(1-t) . \tag{B.2}
\end{equation*}
$$

Then, we expand the following quantity

$$
\begin{equation*}
\left(1-z_{t}\right)=\frac{1}{\rho_{t}}(1-t)\left[1-\frac{\omega_{t}}{R}(1-t)+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right] \tag{B.3}
\end{equation*}
$$

with

$$
\begin{align*}
\rho_{t} & \equiv \frac{a-z^{\prime}+R}{2 R},  \tag{B.4}\\
\omega_{t} & \equiv \frac{2 a^{2}-2 a R-a^{2} z^{\prime}+z^{\prime}}{2\left(a-z^{\prime}+R\right)}, \tag{B.5}
\end{align*}
$$

By using Eq. (A.15), we have

$$
\begin{align*}
\frac{z_{t}}{-\hat{t}}\left(1-z_{t}\right)^{-1+\epsilon} & =\frac{z^{\prime}}{m_{H}^{2} R}\left(\rho_{t}\right)^{-\epsilon}(1-t)^{-1+\epsilon}\left(1-\frac{\omega_{t}}{R}(1-t)+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right)^{-1+\epsilon} \\
& =\frac{z^{\prime}}{m_{H}^{2} R}\left[\left(1-\epsilon \ln \left(\rho_{t}\right)+\frac{1}{2} \epsilon^{2} \ln ^{2}\left(\rho_{t}\right)+\mathcal{O}\left(\epsilon^{2}\right)\right)\right. \\
& \times\left(\frac{1}{\epsilon} \delta(1-t)+\left(\frac{1}{1-t}\right)_{+}+\epsilon\left(\frac{\ln (1-t)}{1-t}\right)_{+}+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)+\mathcal{O}\left(\epsilon^{2}\right)\right) \\
& \times\left(\frac{1}{1-\frac{\omega_{t}}{R}(1-t)} e^{\left.\epsilon \ln \left(1-\frac{\left.\omega_{t}(1-t)\right)}{R}+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right)\right]}\right. \tag{B.6}
\end{align*}
$$

Then, expanding the exponential and computing all the products we get

$$
\begin{align*}
\frac{z_{t}}{-\hat{t}}\left(1-z_{t}\right)^{-1+\epsilon} & =\frac{z^{\prime}}{m_{H}^{2} R}\left\{\frac{1}{\epsilon} \delta(1-t)\right. \\
& +\left[\frac{1}{1-\frac{\omega_{t}}{R}(1-t)}\left(\frac{1}{1-t}\right)_{+}-\ln \left(\rho_{t}\right) \delta(1-t)\right] \\
& +\epsilon\left[\frac{1}{1-\frac{\omega_{t}}{R}(1-t)}\left(\frac{\ln (1-t)}{1-t}\right)_{+}-\frac{\ln \left(\rho_{t}\right)}{1-\frac{\omega_{t}}{R}(1-t)}\left(\frac{1}{1-t}\right)_{+}\right. \\
& +\frac{\ln \left(1-\frac{\omega_{t}}{R}(1-t)\right.}{1-\frac{\omega_{t}}{R}(1-t)}\left(\frac{1}{1-t}\right)_{+}+\frac{1}{2} \delta(1-t) \ln ^{2}\left(\rho_{t}\right) \\
& \left.\left.+\mathcal{O}\left(\epsilon^{2}\right)+\mathcal{O}\left(\frac{(1-t)^{2}}{R}\right)\right]\right\} . \tag{B.7}
\end{align*}
$$

If we directly made use of Eq. (A.15) on the LHS of Eq. (B.6), we would have obtained

$$
\begin{equation*}
\frac{z_{t}}{-\hat{t}}\left(1-z_{t}\right)^{-1+\epsilon}=\frac{z_{t}}{-\hat{t}}\left[\frac{1}{\epsilon} \delta\left(1-z_{t}\right)+\left(\frac{1}{1-z_{t}}\right)_{+}+\epsilon\left(\frac{\ln 1-z_{t}}{1-z_{t}}\right)_{+}+\mathcal{O}\left(\epsilon^{2}\right)\right] . \tag{B.8}
\end{equation*}
$$

By comparing the two RHS we get equations Eq. (3.42,3.43).
The two parameters of Eq. (B.4) and their counterpart with $\hat{t} \rightarrow \hat{u}$ have the same limits as $z^{\prime} \rightarrow 1$.

The last change of variable is the one for $\delta\left(Q^{2}\right)$. Using the standard formula for changing variable in delta function, that is searching for the
zeros of the argument, one finds

$$
\begin{equation*}
\delta\left(Q^{2}\right)=\frac{z^{\prime}}{R} \frac{1}{m_{H}^{2}} \delta(1-t)+(\text { solution with minimum rapidity }) . \tag{B.9}
\end{equation*}
$$

Observe that $Q^{2}=0$ has actually two solution, one is given by the rapidity approaching the maximum value, the other is given by the opposite situation. In Ch. 3 we have split the integration domain exploiting the symmetry due to rapidity, so we can just give the first solution, since the other is never picked up.

## B.1.2 Some other identities

The change of variables we have just proved, that is Eqs. $(3.42,3.43)$, several products between plus distributions and real function appear. The following identities hold

$$
\begin{align*}
& \left(\frac{\ln ^{k}(1-t)}{1-t}\right)_{+} \frac{1}{1-\frac{\omega_{a}}{R}(1-t)}=\left(\frac{\ln ^{k}(1-t)}{1-t}\right)_{+}+\frac{\ln ^{k}(1-t)}{t-1+\frac{R}{\omega_{a}}}  \tag{B.10}\\
& \left(\frac{1}{1-t}\right)_{+} \frac{\ln \left(1-\frac{\omega}{R}(1-t)\right)}{\left.1-\frac{\omega}{R}(1-t)\right)}=\frac{\ln \left(1-\frac{\omega_{a}}{R}(1-t)\right)}{(1-t)\left(1-\frac{\omega_{a}}{R}(1-t)\right)} . \tag{B.11}
\end{align*}
$$

They can be proved by using the representation Eq. (A.14) for plus distribution and the algebraic identity

$$
\begin{equation*}
\frac{1}{1-t} \frac{1}{1-\frac{\omega}{R}(1-t)}=\frac{1}{1-t}+\frac{1}{t-1+\frac{R}{\omega}} . \tag{B.12}
\end{equation*}
$$

The reason why we kept the order $\frac{1-t}{R}$ We are now ready to explain why the expansion of the previous subsection was given up to the order $\frac{1-t}{R}$. Terms are the origin of contributions such as the last addend in Eq. (B.10). At a first sight, since we are interested in the limit $t \rightarrow 1$, it seems a regular contribution to be discarded. But we need to keep in mind that also $z^{\prime} \rightarrow 1$, so that $R \rightarrow 0$. Let $k=0$ and consider the following integral

$$
\begin{equation*}
\int_{t_{\min }}^{1} d t \frac{1}{t-1+R / \omega} \tag{B.13}
\end{equation*}
$$

where $t_{\min }=1 / t_{\max }$ and $t_{\max }$ is given by Eq. (3.28). That is the integral we would compute setting $M=0$ in the Fourier transform. The primitive

## B.2. Other Integrals

is given by a logarithm and the definite integral is given by

$$
\begin{equation*}
\left.\ln \left(t-1+\frac{R}{\omega}\right)\right|_{t_{\min }} ^{1}=\ln \left(\frac{R / \omega}{t_{\min }-1+R / \omega}\right) \tag{B.14}
\end{equation*}
$$

The limit $z^{\prime} \rightarrow 1$ of this logarithm is just a $\ln (2)$.
since there are no more power of $R$ at the denominator, higher orders in $1-t$ in the expansions of the previous sections can be safely neglected.

## B. 2 Other Integrals

In order to assess our change of variable, we can integrate over the rapidity and then consider the limit $z^{\prime} \rightarrow 1$. In other words, we set $M=0$ in Eq. (3.24). The result should reproduce the already known threshold behaviour for $p_{T}$-distributions. For this purpose, the following integral have been used, besides those in App. A.3.1.

Integrals over plus distributions For any plus distribution regularized with endpoint 1 , the following holds

$$
\begin{equation*}
\int_{t_{\min }}^{1} d t(f(t))_{+}=\int_{0}^{t_{\min }} d t f(t) . \tag{B.15}
\end{equation*}
$$

Then, for $t_{\text {min }}=\frac{1}{t_{\max }}$, where $t_{\text {max }}$ is defined in Eq. (3.28),

$$
\begin{align*}
\int_{t_{\text {min }}}^{1} d t\left(\frac{\ln ^{k}(1-t)}{1-t}\right)_{+} & =-\frac{\ln ^{k+1}\left(1-t_{\text {min }}\right)}{k+1} \\
& =-\frac{1}{k+1} \ln ^{k+1}\left(\sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\right)+\mathcal{O}\left(1-z^{\prime}\right), \tag{B.16}
\end{align*}
$$

where we made the expansion for $z^{\prime} \sim 1$

$$
\begin{equation*}
1-t_{\min } \sim \sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}+\mathcal{O}\left((\sqrt{1-z})^{2}\right) \tag{B.17}
\end{equation*}
$$

Integrals over rational logarithmic function Using the following indefinite integral

$$
\begin{equation*}
\int d x \frac{\ln (1-x)}{x-\alpha}=L i_{2}\left(\frac{x-1}{\alpha-x}\right)+\ln (1-x) \ln \left(\frac{\alpha-x}{\alpha-1}\right)+\mathrm{k} . \tag{B.18}
\end{equation*}
$$

it is easy, although algebraically tedious, to integrate functions of the form of the regular remnant of Eq. (B.10).

## B. 3 The $p_{T} \rightarrow 0$ limit

The previous changes of variables can be used if the limit $z^{\prime} \rightarrow 1$ is taken before the limit $p_{T} \rightarrow 0$. We cannot interchange the two limits, as it is explained in section 4.1. Here we present the derivation of the small- $p_{T}$ limit of the cross section. The first observation we make is that in the small $p_{T}$ limit the kinematics approaches the $p_{T}$-integrated one, that is the kinematics of the rapidity distribution.

If we consider the variables Eq. (3.38) and express them in terms of the $z_{1,2}$ variables of Eq. (2.44), we find

$$
\begin{equation*}
z_{t}=z_{1} \frac{(a+1)-2 z_{2}}{2 a-(a+1) z 2} \quad z_{u}=z_{t}\left[z_{1} \leftrightarrow z_{2}\right] . \tag{B.19}
\end{equation*}
$$

As $p_{t} \rightarrow 0, z_{t} \rightarrow z_{1}$ and $z_{u} \rightarrow z_{2}$.

In the small- $p_{T}$ limit, retaining only the most $p_{T}$-divergent part, we find

$$
\left(\frac{1}{1-z_{t}}\right)_{+}\left(\frac{1}{-\hat{t}}\right)^{2}\left(\frac{1}{-\hat{u}}\right)=\frac{1}{\xi_{p}}\left[\frac{1}{m_{H}^{6}} \frac{1}{2} \ln \left(\xi_{p}\right) \delta\left(1-z_{1}\right) \delta\left(1-z_{2}\right)+\mathcal{O}\left(\xi_{p}\right)\right]
$$

$$
\begin{equation*}
\left(\frac{\ln \left(1-z_{t}\right)}{1-z_{t}}\right)_{+}\left(\frac{1}{-\hat{t}}\right)^{2}\left(\frac{1}{-\hat{u}}\right)=\frac{1}{\xi_{p}}\left[\frac { 1 } { m _ { H } ^ { 6 } } \frac { 1 } { 2 } \left(\frac{\pi^{2}}{3}+\ln ^{2}(2)+\frac{1}{4} \ln ^{2}\left(\xi_{p}\right)\right.\right. \tag{B.21}
\end{equation*}
$$

$$
\left.\left.+\ln (2) \ln \left(\xi_{p}\right)\right) \delta\left(1-z_{1}\right) \delta\left(1-z_{2}\right)+\mathcal{O}\left(\xi_{p}\right)\right]
$$

$$
\frac{1}{\hat{t}} \ln \left(\frac{\mu_{F}^{2} z_{t}}{\hat{t}}\right) P_{g g}\left(z_{t}\right)=\frac{1}{\xi_{p}} \frac{1}{m_{H}^{6}}\left[2 N_{c}\left(\frac{1}{1-z_{1}}\right)_{+}+\beta_{0} \delta\left(1-z_{1}\right)\right]
$$

$$
\begin{equation*}
\times\left[\left(\frac{\ln \left(1-z_{2}\right)}{1-z_{2}}\right)_{+}-\frac{1}{8} \ln ^{2}\left(\xi_{p}\right) \delta\left(1-z_{2}\right)\right. \tag{B.22}
\end{equation*}
$$

$$
\left.+\ln \left(\frac{\mu_{F}^{2}}{m_{H}^{2}}\right)\left(\frac{1}{2} \ln \left(\xi_{p}\right) \delta\left(1-z_{2}\right)-\left(\frac{1}{1-z_{2}}\right)_{+}\right)+\mathcal{O}\left(\xi_{p}\right)\right]
$$

$$
\begin{equation*}
\frac{1}{-\hat{t}} p_{g g}\left(z_{t}\right)\left(\frac{\ln \left(1-z_{t}\right)}{1-z_{t}}\right)_{+}=\frac{1}{m_{H}^{2} \xi_{p}} 2 N_{c}\left[\left(\frac{1}{1-z_{2}}\right)_{+}\left(\frac{\ln \left(1-z_{1}\right)}{1-z_{1}}\right)_{+}\right. \tag{B.23}
\end{equation*}
$$

$$
\left.-\frac{1}{2} \ln \left(\xi_{p}\right) \delta\left(1-z_{2}\right)\left(\frac{\ln \left(1-z_{1}\right)}{1-z_{1}}\right)_{+}+\mathcal{O}\left(\xi_{p}\right)\right]
$$

and the case with $t \leftrightarrow u$ is found just swapping $z_{1} \leftrightarrow z_{2}$.

## B. 4 Integrals for the Final Expansions

We need to generalize the results obtained in section [B.2] of this Appendix. We give results outlining the general strategy. The first integral we need is that over plus distributions, as in Eq. (B.16), but with more power of $(1-t)$. Actually, the calculation is very simple in these cases because plus prescriptions are simply removed. For example, one finds that, for $m \geq 1$

$$
\begin{equation*}
\int_{\frac{1}{t_{\max }}}^{1} d t(1-t)^{m}\left(\frac{\ln (1-t)}{1-t}\right)_{+}=\int_{\overline{t_{\max }}}^{1} d t(1-t)^{m-1} \ln (1-t) \tag{B.24}
\end{equation*}
$$

which requires at most an integration by parts. More intricate integrations come from pieces with rational functions which contains logarithms. The most general integral is

$$
\begin{equation*}
\mathcal{I}_{(p, m, n)} \equiv \int_{t_{\text {min }}}^{1} d t \frac{(1-t)^{p} \ln ^{m}(1-t) \ln ^{n}\left(t-1+\frac{R}{\omega}\right)}{t-1+\frac{R}{\omega}} \tag{B.25}
\end{equation*}
$$

with $p=0,1,2$ and we have defined $t_{\min }=\frac{1}{t_{\max }}$. For each value of $p$, the two other parameters assume value as

- $m=n=0$;
- $m=1 \wedge n=0$;
- $m=0 \wedge n=1$;

Again, the case $p=0,1,2 \wedge n=m=0$ can be solved by means of integration by parts or as a special case of the situation we are going to consider.

We define the generating integral

$$
\begin{equation*}
\mathcal{G}_{(p)}(\rho, \eta) \equiv \int_{t_{\min }}^{1} d t(1-t)^{p+\rho}\left(t-1+\frac{R}{\omega}\right)^{\eta-1} \tag{B.26}
\end{equation*}
$$

from which

$$
\begin{equation*}
\mathcal{I}_{(p, m, n)}=\left.\left[\frac{\partial^{m+k}}{\partial \eta^{m} \partial \rho^{k}} \mathcal{G}_{(p, m, n)}(\rho, \eta)\right]\right|_{\eta=\rho=0} \tag{B.27}
\end{equation*}
$$

We write $\left(t-1+\frac{R}{\omega}\right)^{\eta-1}$ as a power series around $t=1$. The $n$-th derivative, with $n \geq 1$ is

$$
\begin{align*}
& \left.\frac{d^{n}}{d t^{n}}\left(t-1+\frac{R}{\omega}\right)^{\eta-1}\right|_{t=1}= \\
& =\left.(-1)^{n}(1-\eta)(2-\eta) \cdots(n-\eta)\left(t-1+\frac{R}{\omega}\right)^{\eta-n-1}\right|_{t=1}  \tag{B.28}\\
& =(-1)^{n} n!\left(1-H(n) \eta+\mathcal{O}\left(\eta^{2}\right)\right)\left(\frac{R}{\omega}\right)^{\eta-n-1},
\end{align*}
$$

where we have used

$$
\begin{align*}
& (1-\eta)(2-\eta) \cdots(n-\eta)=n!+(-\eta)(2 \cdot 3 \cdots n) \\
& +1 \cdot(-\eta) \cdot(3 \cdot 4 \cdots n)+\cdots \\
& +(1 \cdot 2 \cdot 3 \cdots(n-1))(-\eta)+\mathcal{O}\left(\eta^{2}\right) \\
& =n!\cdot\left(1+(-\eta)\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right)+\mathcal{O}\left(\eta^{2}\right)\right)  \tag{B.29}\\
& =n!\cdot\left(1-H(n) \eta+\mathcal{O}\left(\eta^{2}\right)\right)\left(\frac{R}{\omega}\right)^{\eta-n-1},
\end{align*}
$$

being $H(n)$ the $n$-th harmonic number. The integrand is rewritten as

$$
\begin{align*}
& (1-t)^{p+\rho}\left(\frac{R}{\omega}\right)^{\eta-1}+\sum_{n \geq 1}\left(1-H(n) \eta+\mathcal{O}\left(\eta^{2}\right)\right)\left(\frac{R}{\omega}\right)^{\eta-n-1}(1-t)^{\rho+n+p} \\
& =\sum_{n \geq 0}\left(1-H(n) \eta+\mathcal{O}\left(\eta^{2}\right)\right)\left(\frac{R}{\omega}\right)^{\eta-n-1}(1-t)^{\rho+n+p}+\mathcal{O}\left(\eta^{2}\right) \tag{B.30}
\end{align*}
$$

having defined $H(0)=0$, and neglecting ${ }^{1}$ the order $\mathcal{O}\left(\eta^{2}\right)$. Integrating the series term by term, we get

$$
\begin{equation*}
\mathcal{G}_{(p)}(\eta, \rho)=\sum_{n \geq 0}\left[\left(1-H(n) \eta+\mathcal{O}\left(\eta^{2}\right)\right)\left(\frac{R}{\omega}\right)^{\eta-n-1} \frac{\left(1-t_{\min }\right)^{\rho+p+n+1}}{\rho+p+n+1}\right] \tag{B.31}
\end{equation*}
$$

In general, it is not easy to sum this series and, maybe, not even possible in term of known elementary functions. Now, we are going to show that, at least in the two cases we are interested in, this is possible.

$$
\begin{align*}
& \mathbf{m}=\mathbf{0} \text { and } \mathbf{n}=\mathbf{1} \\
& \mathcal{I}_{(p, 0,1)}=\left.\frac{\partial}{\partial \rho} \mathcal{G}_{(p)}(\rho, 0)\right|_{\rho=0} \\
& =\sum_{n \geq 0}\left(\frac{\omega}{R}\right)^{n+1}\left(\frac{\left(1-t_{\text {min }}\right)^{p+n+1} \ln \left(1-t_{\text {min }}\right)}{p+n+1}-\frac{\left(1-t_{\text {min }}\right)^{p+n+1}}{(p+n+1)^{2}}\right) \\
& =\ln \left(1-t_{\text {min }}\right)\left(\frac{R}{\omega}\right)^{p} \sum_{n \geq p+1} \frac{\left(1-t_{\text {min }}\right)^{n}}{n}\left(\frac{\omega}{R}\right)^{n}-\left(\frac{R}{\omega}\right)^{p} \sum_{n \geq p+1} \frac{\left(1-t_{\text {min }}\right)^{n}}{n^{2}}\left(\frac{\omega}{R}\right)^{n} \\
& =\left(\frac{R}{\omega}\right)^{p}\left[\ln \left(1-t_{\text {min }}\right)\left(-\ln \left(1+\frac{\omega}{R}\left(t_{\text {min }}-1\right)\right)-\sum_{n=1}^{p} \frac{\left(1-t_{\text {min }}\right)^{n}}{n}\left(\frac{\omega}{R}\right)^{n}\right)\right. \\
& \left.-L i_{2}\left(\frac{\omega}{R}\left(1-t_{\text {min }}\right)\right)+\sum_{n=1}^{p} \frac{\left(1-t_{\text {min }}\right)^{n}}{n^{2}}\left(\frac{\omega}{R}\right)^{n}\right] \tag{B.32}
\end{align*}
$$

where, in the last line, we have summed the series for $n \geq 1$ and then we have subtracted redundant pieces from $n=1$ up to $n=p$. Moreover, we have used the series representation of the logarithm and the less known one

[^17]of the Di-Logarithmic function.
Threshold expansion of the above integral, for $p=0,1,2$, gives
\[

$$
\begin{align*}
& \mathcal{I}_{(0,0,1)} \sim \ln (2) \ln \left(\sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\right)-\frac{\pi^{2}}{12}+\frac{1}{2} \ln ^{2}(2),  \tag{B.33}\\
& \mathcal{I}_{(1,0,1)} \sim \sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\left(2 \ln (2) \ln \left(\sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\right)-\frac{\pi^{2}}{6}+\ln ^{2}(2)\right), \\
& \mathcal{I}_{(2,0,1)} \sim\left(\frac{a-1}{a+1}\right)\left(1-z^{\prime}\right)\left(4 \ln (2) \ln \left(\sqrt{\frac{a-1}{a+1}} \sqrt{\left(1-z^{\prime}\right)}\right)-\frac{\pi^{2}}{3}+2 \ln ^{2}(2)-\frac{1}{4}\right) . \tag{B.34}
\end{align*}
$$
\]

$\mathrm{m}=1$ and $\mathrm{n}=0$

$$
\begin{align*}
& \mathcal{I}_{(p, 1,0)}=\left.\frac{\partial}{\partial \eta} \mathcal{G}_{(p)}(0, \eta)\right|_{\eta=0} \\
& =\sum_{n \geq 0}\left[-H(n)\left(\frac{\omega}{R}\right)^{n+1} \frac{\left(1-t_{\min }\right)^{p+n+1}}{p+n+1}\right.  \tag{B.36}\\
& \left.+\ln \left(\frac{R}{\omega}\right)\left(\frac{R}{\omega}\right)^{n+1} \frac{\left(1-t_{\min }\right)^{p+n+1}}{p+n+1}\right]
\end{align*}
$$

Performing manipulation similar to those above given, rearranging addends when appropriate, we find

$$
\begin{align*}
\mathcal{I}_{(0,1,0)} \sim & -\frac{1}{2} \ln ^{2}(2)-\ln (2) \ln \left(2 \sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\right)  \tag{B.37}\\
\mathcal{I}_{(1,1,0)} \sim & \sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\left(-5 \ln ^{2}(2)-5 \ln (2)-\frac{1}{8}+\frac{\pi^{2}}{2}\right. \\
& \left.-(1+2 \ln (2)) \ln \left(\sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\right)\right)  \tag{B.38}\\
\mathcal{I}_{(2,0,1)} \sim & \left(\frac{a-1}{a+1}\right)\left(1-z^{\prime}\right)\left(\left(-4 \ln (2)-\frac{5}{2}\right) \ln \left(\sqrt{\frac{a-1}{a+1}} \sqrt{1-z^{\prime}}\right)-6 \ln ^{2}(2)\right. \\
& \left.-6 \ln (2)-\frac{103}{36}\right) . \tag{B.39}
\end{align*}
$$

Eqs.(B.33-B.35) and (B.37-B.39) are all we need to compute corrections to the rapidity integrated cross section near the threshold.

## Appendix C

## Special Functions

In this Appendix we recall some special functions that arise in integrals like those we performed in this thesis.

## C. 1 The Euler Gamma function and its derivatives

The Euler Gamma is defined by the following improper integral

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} d t t^{z-1} e^{-t}, \quad \Re(z)>0 \tag{C.1}
\end{equation*}
$$

Integrating by parts, it is simple to verify the recursion relation

$$
\begin{equation*}
\Gamma(z+1)=z \Gamma(z) \tag{C.2}
\end{equation*}
$$

If evaluated for $z \in \mathbb{R}$, the Gamma function is a real number. Moreover, since $\Gamma(1)=1$ and from Eq.(C.2), it follows that $\Gamma(n+1)=n$ ! for all positive integers.

Splitting the integration region as $[0,1) \cup(1, \infty)$ and evaluating the proper integral after expansion of the exponential, we get the alternative form

$$
\begin{equation*}
\Gamma(z)=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \frac{1}{z+n}+\int_{1}^{\infty} d t t^{z-1} e^{-t} \tag{C.3}
\end{equation*}
$$

Requiring relation in Eq.(C.2) to hold, the Euler Gamma can be analytically extended as a meromorphic function on the complex plane with simple poles on negative integers. Eq.(C.3) is a representation of such extension: the integral converges on the whole complex plane whereas the series explicitly shows the announced simple poles.

We define the DiGamma function as the derivative of the logarithm of the Gamma

$$
\begin{equation*}
\psi_{0}(z):=\frac{d}{d z} \log \Gamma(z) \tag{C.4}
\end{equation*}
$$

From Eq.(C.2) it follows the analogous relation

$$
\begin{equation*}
\psi_{0}(z+1)=\frac{1}{z}+\psi_{0}(z) \tag{C.5}
\end{equation*}
$$

Evaluating the DiGamma for a positive integer $n$ and iterating the last relation, we find

$$
\begin{equation*}
\psi_{0}(n+1)=\psi_{0}(1)+1+\frac{1}{2}+\cdots \frac{1}{n} \tag{C.6}
\end{equation*}
$$

which shows the important link between the DiGamma function and Harmonic numbers.
The value $\psi_{0}(1)=-\gamma_{E}=0.5772 \ldots$ is called Euler-Mascheroni constant. Higher order derivatives give rise to PolyGamma functions

$$
\begin{equation*}
\psi_{n}(z):=\frac{d^{n+1}}{d z^{n+1}} \log \Gamma(z) \tag{C.7}
\end{equation*}
$$

Again, differentiating the recursive relation (C.5), we find the analogous relation for the the $n$ th-order PolyGamma

$$
\begin{equation*}
\psi_{n}(z+1)=\psi_{n}(z)+\frac{(-1)^{n} n!}{z^{n+1}} \tag{C.8}
\end{equation*}
$$

We are particularly interested in asymptotic expansions of the above functions. Let us consider the Laurent expansion of the Gamma function around a generic point $z$

$$
\begin{align*}
\Gamma(z+\varepsilon) & =\Gamma(z)+\Gamma^{\prime}(z) \varepsilon+\frac{1}{2} \Gamma^{\prime \prime}(z) \varepsilon^{2}+\mathcal{O}\left(\varepsilon^{3}\right)  \tag{C.9}\\
& =\Gamma(z)\left(1+\psi_{0}(z) \varepsilon+\frac{1}{2}\left(\psi_{1}(z)+\psi_{0}^{2}(z)\right) \varepsilon^{2}+\mathcal{O}\left(\varepsilon^{3}\right)\right)
\end{align*}
$$

Evaluating for $z=1$ and exploiting relation (C.2) we get the Gamma expansion near the origin

$$
\begin{equation*}
\Gamma(\varepsilon)=\frac{\Gamma(1+\varepsilon)}{\varepsilon}=\frac{1}{\epsilon}-\gamma_{E}+\frac{\varepsilon}{2}\left(\frac{\pi^{2}}{6}+\gamma_{E}^{2}\right)+\mathcal{O}\left(\varepsilon^{2}\right) \tag{C.10}
\end{equation*}
$$

Importantly, as $z \rightarrow \infty$ along the real axis, the DiGamma function shows a different behaviour from other PolyGamma functions. Indeed, the following asymptotics hold

$$
\begin{array}{ll}
\psi_{0}(z) & \approx \log (z)+\mathcal{O}\left(\frac{1}{z}\right) \\
\psi_{n}(z) & \approx \mathcal{O}\left(\frac{1}{z^{n}}\right) \quad n>0 \tag{C.11}
\end{array}
$$

The last two results explain why terms with Gamma derivatives of order higher than the first usually give power suppressed contributions.

## C. 2 Gamma Related Functions

The Beta function is defined by the following integral

$$
\begin{equation*}
B(x, y):=\int_{0}^{1} d t t^{x-1}(1-t)^{y-1} \quad \operatorname{Re} x>0, \text { Re } y>0 \tag{C.12}
\end{equation*}
$$

The Beta is symmetric in its argument and there exists useful representation in terms of the Gamma function

$$
\begin{equation*}
B(x, y)=\frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \tag{C.13}
\end{equation*}
$$

Its partial derivative is given by

$$
\begin{align*}
\frac{\partial}{\partial x} B(x, y) & =B(x, y)\left(\frac{\Gamma^{\prime}(x)}{\Gamma(x)}-\frac{\Gamma^{\prime}(x+y)}{\Gamma(x+y)}\right)  \tag{C.14}\\
& =B(x, y)\left(\psi_{0}(x)-\psi_{0}(x+y)\right)
\end{align*}
$$

The following asymptotic immediately follows from Stirling approximation

$$
\begin{equation*}
B(x, y) \approx \Gamma(x) x^{-y} \quad \text { as } \quad x \rightarrow \infty \text { for fixed } y \tag{C.15}
\end{equation*}
$$

A less straightforward Euler Gamma-related function is represented by the hypergeometric function. We define the $(p, q)$ order hypergeometric function as

$$
\begin{equation*}
{ }_{p} F_{q}\left(a_{1}, a_{2}, \ldots, a_{p} ; b_{1}, b_{2}, \ldots, b_{q} ; z\right):=\sum_{k=0}^{\infty} \frac{\left(a_{1}\right)_{k}\left(a_{2}\right)_{k} \ldots\left(a_{p}\right)_{k}}{\left(b_{1}\right)_{k}\left(b_{2}\right)_{k} \ldots\left(b_{q}\right)_{k}} \frac{z^{k}}{k!} \tag{C.16}
\end{equation*}
$$

where $(q)_{n}$ is called Pochhammer symbol ad its value is

$$
(q)_{n}=\left\{\begin{array}{lr}
1 & \text { if } \quad n=0  \tag{C.17}\\
q(q+1) \cdots(q+n-1) & \text { if } \quad n>0
\end{array}\right.
$$

which is more commonly rewritten as

$$
(q)_{n}\left\{\begin{array}{lll}
1 & \text { if } & n=0  \tag{C.18}\\
\frac{(q+n-1)!}{(q-1)!}=\frac{\Gamma(q+n)}{\Gamma(q)} & \text { if } & n>0
\end{array}\right.
$$

What it is usually called hypergeometric function is the special case

$$
\begin{equation*}
{ }_{2} F_{1}(a, b ; c ; z)=\sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{(c)_{k}} \frac{z^{k}}{k!} \tag{C.19}
\end{equation*}
$$

The hypergeometric function arises in the evaluation of Euler-type integral. Indeed, the following integral representation holds for $\Re(c)>\Re(b)>0$

$$
\begin{equation*}
B(b, c-b)_{2} F_{1}(a, b ; c ; z)=\int_{0}^{1} d t t^{b-1}(1-t)^{c-b-1}(1-z t)^{-a} \tag{C.20}
\end{equation*}
$$

When the upper integration limit is different from one, the related function is said to be incomplete. Unfortunately, incomplete beta or hypergeometric functions do not satisfy as many useful relation as the complete counterparts do. For our purposes, it is sufficient to consider some very recent results, see [23].

First, we define the incomplete Beta function ${ }^{1}$

$$
\begin{equation*}
B_{y}(b, c):=\int_{0}^{y} d x x^{b-1}(1-x)^{c-1} \tag{C.21}
\end{equation*}
$$

which satisfies the following representation

$$
\begin{equation*}
B_{y}(b, c)=\frac{y^{b}}{b}{ }_{2} F_{1}(b, 1-c ; 1+b ; y) \tag{C.22}
\end{equation*}
$$

We now introduce the incomplete Pochhammer Ratio

$$
\begin{equation*}
[b, c ; y]_{n}:=\frac{B_{y}(b+n, c-b)}{B(b, c-b)} \tag{C.23}
\end{equation*}
$$

The limit $[b, c ; y]_{n} \rightarrow(b)_{n} /(c)_{n}$ as $y \rightarrow 1$ immediately follows from identity (C.13).

We can now define the Incomplete Hypergometric Function

$$
\begin{equation*}
{ }_{2} F_{1}(a,[b, c ; y] ; x):=\sum_{n=0}^{\infty}(a)_{n}[b, c ; y]_{n} \frac{x^{n}}{n!} \quad \text { with } 0 \leq y \leq 1 \tag{C.24}
\end{equation*}
$$

which possesses the following integral representation

$$
\begin{equation*}
{ }_{2} F_{1}(a,[b, c ; y] ; x)=\frac{1}{B(b, c-b)} \int_{0}^{y} d t t^{b-1}(1-t)^{c-b-1}(1-x t)^{-a} \tag{C.25}
\end{equation*}
$$

The objects introduced above and their properties should be sufficient for the purposes of this thesis.

[^18]APPENDIX C. SPECIAL FUNCTIONS

## Bibliography

[1] Guido Altarelli and Giorgio Parisi. "Asymptotic freedom in parton language". In: Nuclear Physics B 126.2 (1977), pp. 298-318.
[2] Charalampos Anastasiou et al. "Dilepton Rapidity Distribution in the Drell-Yan Process at Next-to-Next-to-Leading Order in QCD". In: Physical Review Letters 91.18 (Oct. 2003). ISSN: 1079-7114. Doi: 10.1103/physrevlett.91.182002. URL: http://dx.doi.org/10. 1103/PhysRevLett.91.182002.
[3] Pulak Banerjee et al. "Threshold resummation of the rapidity distribution for Drell-Yan production at NNLO+NNLL". In: Physical Review D 98.5 (2018). ISSN: 2470-0029. DOI: 10.1103/physrevd.98.054018. URL: http://dx.doi.org/10.1103/PhysRevD.98.054018.
[4] Pulak Banerjee et al. "Threshold resummation of the rapidity distribution for Higgs production at NNLO+NNLL". In: Physical Review D 97.5 (Mar. 2018). ISSN: 2470-0029. DOI: 10.1103/physrevd. 97. 054024. URL: http://dx.doi.org/10.1103/PhysRevD.97.054024.
[5] Paolo Bolzoni, Stefano Forte, and Giovanni Ridolfi. "Renormalization group approach to Sudakov resummation in prompt photon production". In: Nuclear Physics $B$ 731.1-2 (Dec. 2005), pp. 85-108. ISSN: 0550-3213. DOI: $10.1016 / \mathrm{j}$. nuclphysb .2005 .07 .036. URL: http: //dx.doi.org/10.1016/j.nuclphysb.2005.07.036.
[6] Roberto Bonciani et al. "Sudakov resummation of multiparton QCD cross sections". In: Physics Letters B 575.3-4 (Nov. 2003), pp. 268278. ISSN: 0370-2693. DOI: $10.1016 / \mathrm{j}$. physletb. 2003.09.068. URL: http://dx.doi.org/10.1016/j.physletb.2003.09.068.
[7] Giuseppe Bozzi et al. "Higgs boson production at the LHC: Transversemomentum resummation and rapidity dependence". In: Nuclear Physics B 791.1-2 (Mar. 2008), pp. 1-19. ISSN: 0550-3213. DOI: $10.1016 / \mathrm{j}$.
nuclphysb.2007.09.034. URL: http://dx.doi.org/10.1016/j. nuclphysb. 2007.09.034.
[8] Giuseppe Bozzi et al. "Transverse-momentum resummation and the spectrum of the Higgs boson at the LHC". In: Nuclear Physics B 737.1-2 (Mar. 2006), pp. 73-120. ISSN: 0550-3213. DOI: $10.1016 / \mathrm{j}$. nuclphysb.2005.12.022. URL: http://dx.doi.org/10.1016/j. nuclphysb. 2005.12.022.
[9] S Catani and L Trentadue. "Resummation of the QCD perturbative series for hard processes". In: Nuclear Physics B 327.2 (1989), pp. 323352.
[10] John C Collins, Davison E Soper, and George Sterman. "Transverse momentum distribution in Drell-Yan pair and W and Z boson production". In: Nuclear physics B 250.1-4 (1985), pp. 199-224.
[11] Yuri L Dokshitzer. "Calculation of the structure functions for deep inelastic scattering and $\mathrm{e}+\mathrm{e}-$ annihilation by perturbation theory in quantum chromodynamics". In: Zh. Eksp. Teor. Fiz 73 (1977), p. 1216.
[12] R Keith Ellis, W James Stirling, and Bryan R Webber. QCD and collider physics. Cambridge university press, 2003.
[13] Ludvig D Faddeev and Victor N Popov. "Feynman diagrams for the Yang-Mills field". In: Physics Letters B 25.1 (1967), pp. 29-30.
[14] Daniel de Florian, Anna Kulesza, and Werner Vogelsang. "Threshold resummation for high-transverse-momentum Higgs production at the LHC". In: Journal of High Energy Physics 2006.02 (Feb. 2006), pp. 047-047. ISSN: 1029-8479. DOI: $10.1088 / 1126-6708 / 2006 / 02 /$ 047. URL: http://dx.doi.org/10.1088/1126-6708/2006/02/047.
[15] Stefano Forte and Giovanni Ridolfi. "Renormalization group approach to soft gluon resummation". In: Nuclear Physics B 650.1-2 (Feb. 2003), pp. 229-270. ISSN: 0550-3213. DOI: $10.1016 /$ s0550-3213(02)010349. URL: http://dx.doi.org/10.1016/S0550-3213(02)01034-9.
[16] Christopher J Glosser and Carl R Schmidt. "Next-to-leading Corrections to the Higgs Boson Transverse Momentum Spectrum in Gluon Fusion". In: Journal of High Energy Physics 2002.12 (Dec. 2002), pp. 016-016. ISSN: 1029-8479. DOI: $10.1088 / 1126-6708 / 2002 / 12 /$ 016. URL: http://dx.doi.org/10.1088/1126-6708/2002/12/016.
[17] Vladimir Naumovich Gribov and Lev N Lipatov. DEEP INELASTIC ep-SCATTERING IN A PERTURBATION THEORY. Tech. rep. Inst. of Nuclear Physics, Leningrad, 1972.
[18] Toichiro Kinoshita. "Mass singularities of Feynman amplitudes". In: Journal of Mathematical Physics 3.4 (1962), pp. 650-677.
[19] Tsung-Dao Lee and Michael Nauenberg. "Degenerate systems and mass singularities". In: Physical Review 133.6B (1964), B1549.
[20] Gillian Lustermans, Johannes K. L. Michel, and Frank J. Tackmann. Generalized Threshold Factorization with Full Collinear Dynamics. 2019. arXiv: 1908.00985 [hep-ph].
[21] Claudio Muselli, Stefano Forte, and Giovanni Ridolfi. "Combined threshold and transverse momentum resummation for inclusive observables". In: Journal of High Energy Physics 2017.3 (Mar. 2017). IsSN: 10298479. DOI: 10.1007/jhep03(2017) 106. URL: http://dx.doi.org/ 10.1007/JHEP03(2017) 106.
[22] Taizo Muta. Foundations of quantum chromodynamics: an introduction to perturbative methods in gauge theories. world scientific, 1998.
[23] Mehmet Ali Ozarslan and Ceren Ustaog Lu. In: Mathematics 2019, 7, 483 ().
[24] M. Spira et al. "Higgs boson production at the LHC". In: Nuclear Physics B 453.1 (1995), pp. 17-82. ISSN: 0550-3213. DOI: https:// doi.org/10.1016/0550-3213(95) 00379-7. URL: https://www. sciencedirect.com/science/article/pii/0550321395003797.
[25] George Sterman. "Summation of large corrections to short-distance hadronic cross sections". In: Nuclear Physics B 281.1 (1987), pp. 310364. ISSN: 0550-3213. DOI: https : / / doi . org / 10 . 1016 / 0550-3213(87)90258-6. URL: https://www.sciencedirect.com/science/ article/pii/0550321387902586.
[26] Kenneth G. Wilson and J. Kogut. "The renormalization group and the expansion". In: Physics Reports 12.2 (1974), pp. 75-199.


[^0]:    ${ }^{1}$ In the following we will use the more common parameter $\alpha_{s}=\frac{g_{s}^{2}}{4 \pi}$
    ${ }^{2}$ In a certain way, these are the most important features of a Lie group. Indeed, given a representation of the algebra, if the exponential map is surjective, which is the case of $S U(n)$, then we also have a group representation.

[^1]:    ${ }^{3}$ Actually, it is also large if compared weak bosons and, importantly, the Higgs Boson.

[^2]:    ${ }^{4}$ The procedure is indeed called gauge fixing

[^3]:    ${ }^{5}$ Observe that the covariant derivative, which appears in the ghost term, is in the adjoint representation. Since structure constants are all zero in abelian theories, ghosts decouple from gauge fields in that case. They remain as free propagators that can be ignored. This is the case of electromagnetic interactions.

[^4]:    ${ }^{6}$ We call tree level, or Born level, the first non trivial perturbative order.
    ${ }^{7}$ Whenever the series is truncated, a spurious $\mu$ dependence remains in the last perturbative order added.

[^5]:    ${ }^{8}$ Note that also Eq.(1.21) does not hold anymore, because it is a perturbative solution

[^6]:    ${ }^{9} \mathrm{Or}$ computed with other techniques, for example studying QCD on the lattice.
    ${ }^{10}$ Of course the parton distribution of a proton will be different to that of neutron, for example.

[^7]:    ${ }^{11}$ If the mass is not zero, collinearity is precluded by the on-shellness condition for the radiated real particle.

[^8]:    ${ }^{12}$ The first prediction for Drell-Yan was obtained within the naive Parton Model and discrepancies with data were very large (about a factor two in total cross section).It was only after QCD was formulated and its effects were included that the prediction drastically improved.

[^9]:    ${ }^{13}$ Assuming the parton model still being true.

[^10]:    ${ }^{1}$ constant in the large N limit

[^11]:    ${ }^{2}$ Actually, for LL prediction it would be sufficient to determine the $g_{12}$ numerical factors.

[^12]:    ${ }^{3}$ The authors of this paper try to go deeper in resummation theory, suggesting a technique for dealing consistently with both, threshold and transverse momentum resummation.

[^13]:    ${ }^{4}$ We choose here, for simplicity, the case of Drell-Yan.

[^14]:    ${ }^{5}$ The hard scale, the renormalization and the factorization scales need to be set equal.

[^15]:    ${ }^{1}$ This is absolutely general and follows from the conservation of the total momentum.

[^16]:    ${ }^{1}$ At least continuity over $[0,1]$ is ensured because the test function is supposed to be regular.

[^17]:    ${ }^{1}$ This is sufficient only for our calculation.

[^18]:    ${ }^{1}$ Note that it is no more symmetric in its argument.

