## Matching Threshold and

## Transverse Momentum

 Resummations
## Bachelor Degree in Physics

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DI MILANO

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#### Abstract

:

In perturbative quantum chromodynamics it is possible to sum to all perturbative orders contributions to inclusive differential cross-sections for processes such as Higgs or gauge boson production that correspond to the emission of soft (i.e. small-energy) gluons. These contributions are dominant in two specific kinematic limits: the limit in which the energy for production of the boson approaches threshold, and the limit in which its transverse momentum approaches zero. In this thesis, we study these contributions in the simultaneous threshold and small transverse-momentum limit and we investigate issues related to the possible non-commutativity of the two.


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Academic year : 2022/2023

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## Preface

Quantum Chromodynamics is the fundamental theory describing the strong interaction, one of the four fundamental forces of nature. It is responsible for binding together quarks and gluons, allowing for the existence of protons and neutrons as well as all other hadrons. For these reasons, understanding properly quantum chromodynamics is essential in collider physics to make accurate previsions for observables and seek for eventual evidence of physics beyond the Standard Model.

In the realm of quantum field theory, physical quantities are computed using perturbation theory. However, the presence of non-abelian dynamics in Quantum Chromodynamics renders these calculations considerably more challenging than their counterparts in Quantum Electrodynamics. In this context, all physical observables can be expressed as series expansions in powers of a coupling constant $\alpha_{s}$, whose value, unlike QED or $\phi^{4}$ theory, logarithmically decreases with energy. Consequently, the theory is characterized by an energy scale around the GeV range, beyond which truncating the expansion at a certain order yields reasonably accurate approximations for the observables.

However, in situations involving low-energy predictions, the reliability of the results necessitates knowledge of an increasing number of orders in the expansion. Moreover, certain processes under study may involve multiple energy scales. The ratios of these scales, or, more commonly, their logarithms, can then give rise to a large dimensionless parameter that hampers the convergence of the perturbative series. Hence, all its contributions have to be resummed at all orders. This is the case either of significantly larger scales than the characteristic one, requiring high-energy resummation, or of scales considerably smaller, which necessitates a soft one, with the latter forming the primary focus of this thesis.

In this context, the two most significant situations are the ones due to the presence of soft energy or small transverse momentum scales.

Over the past two decades, considerable effort has been dedicated to developing and refining specific resummation techniques suitable for these types of logarithms. One such approach is the conjugate space method, which aims to factorize out from the observables these logarithmic contributions when performing a particular integral transformation. Subsequently, these contributions are resummed at all orders using the same renormalization group argument that allowed the construction of the perturbative series in the first place.

However, resummations of soft and collinear logarithms can lead to different results, even in regions where both logarithms are large, such as near the threshold and at small transverse momentum. The shared origin of these contributions from singularities in diagrammatic calculations does not guarantee, a priori, that the two limits in which each of them is large necessarily commute, as they are associated with approaching a singular region from different directions. In this thesis, we will perform fixed-order calculations for threshold and transverse momentum-resummed observables to investigate the potential commutation of these two kinematic limits.

In the first chapter, we will provide a brief overview of the fundamental steps involved in the transition from ordinary quantum mechanics to the formulation of quantum chromodynamics. It is important to note that the exposition in this chapter is not exhaustive but aims to give an introductory flavor of the subject to the inexperienced reader. Additionally, we will discuss some aspects of phenomenology and the origin of the singularities to help establish a foundation
for further exploration.
The second chapter, instead, delves into the underlying ideas and techniques behind resummation of soft and collinear singularities, starting from phase space factorization up to the actual resummation in conjugate space.

In the third, explicit fixed order calculation will be carried out, showing the non commutation of the two limits. Furthermore, an attempt is made to derive the exact functional relationship that links the two expressions.

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Inveruno, July 2023

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## A theory for strong interactions

To effectively address the task of this thesis, which focuses on advanced topics in perturbative QCD, we will begin by offering a brief overview of the key steps that motivate the investigation of this subject. Without any claim of exhaustivity and aiming only to provide a general idea of the subject to the less experienced reader, we take inspiration from the works of $[1-4]$ and start in Sec. 1.1 by introducing some naive motivations to the fundamental principles of quantum field theory (QFT) as a natural extension of quantum mechanics (QM). Subsequently, in Sec. 1.2 and 1.3 , we will explore how this new framework can effectively describe particle physics, specifically focusing on strong interactions. Finally, in Sec. 1.4, 1.5, and 1.6, we will introduce crucial aspects of this theory that underscore the necessity for a resummation formalism, which will be expanded in the following chapter.

### 1.1. The need for fields

It is well-known that microscopical world is efficiently described by quantum mechanics. This new theory not only provided a satisfactory model for the atomic spectra, but also fired up a true revolution in the world of physics which led to abandon the classical notion of path on behalf of that of wave function, a probability distribution of positions in space.

However, when addressing the challenge of describing the dynamics of elementary particles, QM needs to incorporate the fundamental principles of special relativity (SR). This requirement entails considering a multitude of complications and cannot be achieved simply by setting the energy eigenvalues of a free particle as $E(\vec{p})=\sqrt{p^{2}+m^{2}}$ instead of $E(\vec{p})=\frac{p^{2}}{2 m}$. Indeed, if we calculate the probability amplitude for a free particle thus described to travel beyond its light-cone (i.e. to travel from the origin of space-time to a certain position $\vec{x}$ in a time $t$ that would require exceeding the speed of light limit), we will find that:

$$
\begin{aligned}
\langle\vec{x}| e^{-i \hat{H} t}|\overrightarrow{0}\rangle & =\int d^{3} \vec{p}\langle\vec{x}| e^{-i \hat{H} t}|\vec{p}\rangle\langle\vec{p} \mid \overrightarrow{0}\rangle \\
& =\int \frac{d^{3} \vec{p}}{(2 \pi)^{3}} e^{i p \cdot x-i E_{\vec{p}} t} \\
& =2 \pi \int_{0}^{\infty} \frac{p^{2} d p}{(2 \pi)^{2}} \int_{-1}^{1} d \xi e^{i p x \xi-i E_{\vec{p}} t} \\
& =\frac{-i}{(2 \pi)^{2} x} \int_{-\infty}^{\infty} d p p e^{i p x} e^{-i t \sqrt{p^{2}+m^{2}}}
\end{aligned}
$$

$$
\begin{align*}
& =\frac{-i}{(2 \pi)^{2} x} \int_{m}^{\infty} d(i z) i z e^{-z x}\left(e^{t \sqrt{z^{2}-m^{2}}}-e^{-t \sqrt{z^{2}-m^{2}}}\right) \\
& =\frac{i}{2 \pi^{2} x} e^{-m x} \int_{m}^{\infty} d z z e^{-(z-m) x} \sinh \left(t \sqrt{z^{2}-m^{2}}\right) \tag{1.1}
\end{align*}
$$

whose square modulus is blatantly non zero, in open contradiction with the experimental evidence of the impossibility of superluminal travel.

A proper relativistic extension of quantum mechanics can instead be obtained through the promotion of time to a operator and the introduction of a "fifth time" parametrizing the now four space-time operators as in ref. [5]. However, the historically most popular option is represented by the inverse process, where spatial coordinates, like time, are treated as parameters and Lorentz invariance is imposed on the time evolution operator. In the interaction picture, this equals to the requirement that

$$
\begin{equation*}
\hat{U}_{I}\left(t_{0}, t\right)=\mathcal{T} \exp \left\{-i \int_{t_{0}}^{t} d t^{\prime} \hat{V}\left(t^{\prime}\right)\right\} \tag{1.2}
\end{equation*}
$$

has to be invariant under Minkowski spacetime isometries, i.e. under transformations of the type

$$
\begin{equation*}
x \rightarrow \hat{L}(\Lambda, a) x=\Lambda x+a \tag{1.3}
\end{equation*}
$$

where $a$ is a 4 -dimensional vector and $\Lambda$ is a $4 \times 4$ dimensional matrix representing a generic Lorentz transformation ${ }^{1}$. To satisfy this requirement, we can consider the hypothesis where $\hat{V}(t)$ is an integral over the positions space of a certain interaction density:

$$
\begin{equation*}
\hat{V}(t)=\int d^{3} \vec{x} \mathscr{H}(\vec{x}, t) \tag{1.4}
\end{equation*}
$$

where $\mathscr{H}(x)$ is a scalar in the sense $\hat{L}(\Lambda, a) \mathscr{H}(x) \hat{L}^{\dagger}(\Lambda, a)=\mathscr{H}(\Lambda x+a)$. In this way $\hat{U}_{I}\left(t_{0}, t\right)$ becomes an integral over space-time and we can expand it as:

$$
\begin{equation*}
\hat{U}_{I}\left(t_{0}, t\right)=1+\sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int d^{4} x_{1} \cdots d^{4} x_{n} \mathcal{T}\left[\mathscr{H}\left(x_{1}\right) \cdots \mathscr{H}\left(x_{n}\right)\right] \tag{1.5}
\end{equation*}
$$

which is a definite integral on spacetime and then manifestly invariant except for the timeordered product of operators. If the scalar $\mathscr{H}(x)$ does not commute at space (or light)-like separations, the application of the isometry might end up changing the operator product in the integral, spoiling invariance. For this reason, we require:

$$
\begin{equation*}
\left[\mathscr{H}(x) ; \mathscr{H}\left(x^{\prime}\right)\right]=0 \quad \text { for } \quad\left(x-x^{\prime}\right)^{2} \geq 0 \tag{1.6}
\end{equation*}
$$

It's now time to ask ourselves what can be a possible form of $\mathscr{H}(x)$. In our setting, spacetime coordinates are only parameters, thus we can no more decompose operators in terms of them. However, another effective decomposition can be achieved through the means of second quantization formalism which allows us to write any operator in terms of sums of products of annihilation and creation operators:

$$
\begin{equation*}
\hat{V}=\sum_{N} \sum_{i_{1} \cdots i_{N}} \sum_{j_{1} \cdots j_{N}} \mathcal{V}_{i_{1} \cdots i_{N}, j_{1} \cdots j_{N}} a_{i_{1}}^{\dagger} \cdots a_{i_{N}}^{\dagger} a_{j_{N}} \cdots a_{j_{1}} \tag{1.7}
\end{equation*}
$$

[^0]where operators $a_{i}$ and $a_{j}^{\dagger}$ satisfy canonical (anti)commutations relations ${ }^{2}$ and destroy/create a certain state of a complete basis of the Hilbert space labelled by the indexes $\left\{i_{k}\right\},\left\{j_{l}\right\}$. This particular choice allows us to a very natural definition of the interaction density in terms of annihilation and creation field operators:
\[

$$
\begin{align*}
& \hat{V}=\sum_{N} \sum_{i_{1} \cdots i_{N}} \sum_{j_{1} \cdots j_{N}} \int d^{3} x \tilde{\mathcal{V}}_{i_{1} \cdots i_{N}, j_{1} \cdots j_{N}} \psi_{i_{1}}^{+}(x) \cdots \psi_{i_{N}}^{+}(x) \psi_{j_{N}}^{-}(x) \cdots \psi_{j_{1}}^{-}(x) \\
& \text { where } \quad a_{j}=\sum_{i} \int d^{3} x u_{i j}(x) \psi_{i}^{-}(x) \text { and } a_{j}^{\dagger}=\sum_{i} \int d^{3} x u_{i j}^{*}(x) \psi_{i}^{+}(x) \tag{1.8}
\end{align*}
$$
\]

The interaction density obtained in this way can be further adjusted to satisfy the "causality" condition given by Eq. (1.6). This can be achieved by constructing new field operators $\psi(x)$ and $\psi^{\dagger}(x)$ as linear combinations of the previous operators $\psi^{-}(x)$ and $\psi^{+}(x)$ in a manner that ensures

$$
\begin{equation*}
\left[\psi_{i}(x) ; \psi_{j}\left(x^{\prime}\right)\right]=\left[\psi_{i}(x) ; \psi_{j}^{\dagger}\left(x^{\prime}\right)\right]=0 \quad \text { for } \quad\left(x-x^{\prime}\right)^{2} \geq 0 \tag{1.9}
\end{equation*}
$$

In light of special relativity then, a quantum theory formulated in terms of fields represents the most natural extension of quantum mechanics. Within this formalism, the states belonging to a specific Hilbert space and characterized by a fixed number of particles are replaced by field operators that exist in a broader Fock space, which is indeed the direct sum of all possible Hilbert space. In this expanded framework, particles have the ability to be created and annihilated, allowing for a more comprehensive description of the world of high energy physics.

### 1.2. Fields and particles

The condition of a scalar $\mathscr{H}(x)$, though implying that the product of these field operators is required to be scalar, imposes no restriction on the fields themselves, which, in fact, can transform according to any conceivable representation of the Lorentz group.

A group representation enables us to describe elements of an abstract group $G$, in our case the group of Lorentz transformation, by utilizing the potential automorphisms of an n-dimensional vector space $V$. These automorphisms can range from the straightforward case of the identity to more intricate transformations, as elaborated upon later, gaining a powerful framework for understanding the behavior of different objects under the action of the same group of transformations. A subspace $W \in V$ is called $G$-invariant if it is sent into itself under the action of some representation $\rho$ of the group $G$, with the restriction of $\rho$ to such a subspace then called subrepresentation. When a representation $\rho$ does not possess any nontrivial subrepresentations, it is referred to as an irreducible representation. The significance of this classification lies in the fact that any general representation can then be decomposed into a combination of irreducible representations, allowing for a deeper understanding of the underlying structure and dynamics of the original one, as it provides insights into the fundamental building blocks from which it is constructed.

It is then natural to break down the fields defined in the previous section, which can generally transform according to particularly intricate representations of the Lorentz group, into irreducible fields that transform according to only one irreducible representation. By doing so, we can identify, among fields containing different species of particles (i.e., transforming according to different rules), the ones containing particles all behaving likewise. The particular choice of $\rho$ will then give rise to the scalar, spinor, vector particles that are observed in experimental particle physics.

[^1]In order to determine the dynamics of such fields we want to derive some field equations using the stationary action principle where the action is defined as a functional of the field $\psi$ :

$$
\begin{equation*}
\mathscr{S}[\psi]=\int d t L[\psi(t), \dot{\psi}(t)] . \tag{1.10}
\end{equation*}
$$

Similarly to what we have done with the time evolution operator, we want $\mathscr{S}$ to be relativistically invariant, so we can write the Lagrangian $L$ in terms of a certain scalar Lagrangian density, such that:

$$
\begin{equation*}
\mathscr{S}[\psi]=\int d^{4} x \mathscr{L}\left[\psi(x), \partial_{\mu} \psi(x)\right] \tag{1.11}
\end{equation*}
$$

From variational principles then, the choice of $\psi$ which minimizes this functional obeys a set of Euler-Lagrange field equations:

$$
\begin{equation*}
\frac{\partial}{\partial x^{\mu}} \frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu} \psi_{j}\right)}=\frac{\partial \mathscr{L}}{\partial \psi_{j}} \tag{1.12}
\end{equation*}
$$

where $\psi_{j}$ is the $j$-th component of the field $\psi$.
We will now write Lagrangian densities as the most general scalar combination of fields $\psi$, $\psi^{\dagger}$ and their derivatives for different types of irreducible representations. In the trivial case of a scalar representation it is immediate for a free scalar, in general, complex field to write the Lagrangian as a combination of the kind

$$
\begin{equation*}
\mathscr{L}\left[\psi(x), \partial_{\mu} \psi(x)\right]=-\frac{1}{2} \partial^{\mu} \psi^{\dagger}(x) \partial_{\mu} \psi(x)-\frac{1}{2} m^{2} \psi^{\dagger}(x) \psi(x) . \tag{1.13}
\end{equation*}
$$

Applying Euler-Lagrange equation we can then observe that this particular kind of field evolves accordingly to the well-known Klein-Gordon equation:

$$
\begin{equation*}
\left(\square+m^{2}\right) \psi(x)=0 . \tag{1.14}
\end{equation*}
$$

To take in account for interactions, we may add other scalar combinations to Lagrangian that can be in the form of Eq. (1.4) or, for vector sources, of a derivative coupling, obtaining:

$$
\begin{equation*}
\mathscr{L}\left[\psi(x), \partial_{\mu} \psi(x)\right]=-\frac{1}{2} \partial^{\mu} \psi^{\dagger}(x) \partial_{\mu} \psi(x)-\frac{1}{2} m^{2} \psi^{\dagger}(x) \psi(x)-J^{\mu} \partial_{\mu} \psi-\mathscr{H}(x) \tag{1.15}
\end{equation*}
$$

which is the Lagrangian density for an interacting scalar field.
Another straightforward example is provided by vector fields $A^{\mu}(x)$, where the representation is based on the same $4 \times 4$ dimensional matrices as mentioned in Eq. (1.3). These vector fields encompass descriptions of photons, gluons, as well as the bosons $W$ and $Z$ and their Lagrangian can be written in terms of the most general scalar combinations of fields and their derivatives:

$$
\begin{equation*}
\mathscr{L}\left[A, \partial_{\mu} A\right]=-\frac{1}{2} \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu}+\frac{1}{2} \partial_{\mu} A_{\nu} \partial^{\nu} A^{\mu}-\frac{1}{2} m^{2} A^{\mu} A_{\mu} \tag{1.16}
\end{equation*}
$$

We can then conveniently define an antisymmetric rank-2 tensor $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$ (which in the electromagnetic case corresponds to the Faraday tensor) to more compactly write the Lagrangian and add a coupling to a certain four-current $J^{\mu}$

$$
\begin{equation*}
\mathscr{L}\left[A, \partial_{\mu} A\right]=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{2} m^{2} A^{\mu} A_{\mu}-J^{\mu} A_{\mu} \tag{1.17}
\end{equation*}
$$

In this setting, field equations are given by

$$
\begin{equation*}
\partial_{\lambda} F^{\lambda \mu}+m^{2} A^{\mu}=-J^{\mu} \tag{1.18}
\end{equation*}
$$

which resemble Maxwell equations with the addition of a mass term $m^{2} A^{\mu}$. If we consider a theory of gauge invariant vector fields, such as Electromagnetism, we expect the Lagrangian density to be invariant under gauge transformations of the type:

$$
\begin{equation*}
A_{\mu}(x) \longrightarrow A_{\mu}(x)+\frac{1}{q} \partial_{\mu} \omega(x) \tag{1.19}
\end{equation*}
$$

We observe that the mass term is manifestly non invariant, since $m^{2} A^{\mu} \rightarrow m^{2}\left(A_{\mu}(x)+\right.$ $\left.q^{-1} \partial_{\mu} \omega(x)\right)$. It then follows that a gauge invariant vector field theory has to be massless and its fields evolve accordingly to Maxwell equations.

A general description for fermions can then be given by the Dirac representation of the Lorentz group. This representation is generated by the set of matrices:

$$
\begin{equation*}
S^{\mu \nu}=-\frac{i}{4}\left[\gamma^{\mu} ; \gamma^{\nu}\right] \quad \text { where } \quad\left\{\gamma^{\mu} ; \gamma^{\nu}\right\}=2 \eta^{\mu \nu} . \tag{1.20}
\end{equation*}
$$

In the three-dimensional Euclidean case, we can see that if we choose these Dirac $\gamma$ matrices as the usual Pauli $\sigma$ matrices, it turns out that the generators are in the form:

$$
\begin{equation*}
S^{i j}=\frac{1}{2} \epsilon^{i j k} \sigma^{k} \tag{1.21}
\end{equation*}
$$

which is the common two-dimensional representation of the group of three-dimensional rotation used in QM to describe spin- $\frac{1}{2}$ particles. A convenient extension to the four-dimensional spacetime is given by the choice:

$$
\gamma^{\mu}=\left[\begin{array}{cc}
0 & \sigma^{\mu}  \tag{1.22}\\
\bar{\sigma}^{\mu} & 0
\end{array}\right] \Longrightarrow S^{0 i}=-\frac{i}{2}\left[\begin{array}{cc}
\sigma^{i} & 0 \\
0 & -\sigma^{i}
\end{array}\right], \quad S^{i j}=\frac{1}{2} \epsilon^{i j k}\left[\begin{array}{cc}
\sigma^{k} & 0 \\
0 & -\sigma^{k}
\end{array}\right]
$$

where $\sigma=(\mathbb{I}, \vec{\sigma})$ and $\bar{\sigma}=(\mathbb{I},-\vec{\sigma})$. There, it is worth noting that the elements $S^{0 i}$ are antiHermitian. Consequently, the spinor representation $\Lambda_{1 / 2}=\exp \left(-i / 2 \omega_{\mu \nu} S^{\mu \nu}\right)$ is non-unitary, which has implications for the product $\psi^{\dagger} \psi$ that no longer behaves as a scalar.

To construct a scalar quantity $\bar{\psi} \psi$ that can be included in the Lagrangian then, we need to find a field $\bar{\psi}$ that transforms as $\bar{\psi} \rightarrow \bar{\psi} \Lambda_{1 / 2}^{-1}$. By exploiting the (anti)commutation properties between $S^{\mu \nu}$ and $\gamma^{0}$ under the condition that $S^{\mu \nu}$ is (anti)unitary, we can choose $\bar{\psi} \equiv \psi^{\dagger} \gamma^{0}$. Therefore, we can express the Lagrangian for free Dirac spinor fields as follows

$$
\begin{equation*}
\mathscr{L}\left[\psi(x), \partial_{\mu} \psi(x)\right]=\bar{\psi}\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi \equiv \bar{\psi}(i \not \partial-m) \psi \tag{1.23}
\end{equation*}
$$

whose field equations correspond to the well-known Dirac equation ${ }^{3}$ :

$$
\begin{equation*}
(i \not \partial-m) \psi=0 \tag{1.24}
\end{equation*}
$$

[^2]
### 1.3. Quantum Chromodynamics

It is now time to formulate a theory capable of describing the phenomena of strong interactions, which serves as the principal environment of this thesis.

From observations in experimental particle physics we know that a large class of particles like protons, neutrons, pions, etc. are made up of fermions called quarks bounded together by some kind of interaction which is extremely strong at short distances, while becomes negligible at larger scales.

While the electromagnetic force is responsible for the binding of electrons and protons in atoms, the situation is more intricate when it comes to quarks. In fact, quarks can form neutral bound states referred to as hadrons, which can be composed of either two quarks (mesons) or three quarks (baryons).


Figure 1.1: Quark structure of two different colourless hadrons. On the left, a pion made up by a blue $u$ and an anti-blue $\bar{d}$ quark, on the right a proton formed by the combination of a red $u$, a green $d$ and a blue $u$.

We can then imagine strong interaction as a sort of a modified electromagnetism, where, instead of one, all the quarks can assume values in a space of three different charges called colours, in analogy with the fact that both the combinations of the three primary colours and two complementary ones can give a neutral white state.

We know that in Electrodynamics, fields are invariant under gauge transformations of the kind Eq. (1.19). This equals, for fermions, to require the invariance under the following transformations:

$$
\begin{array}{r}
\psi \rightarrow e^{i \omega(x)} \psi \\
\bar{\psi} \rightarrow e^{-i \omega(x)} \bar{\psi} \tag{1.25}
\end{array}
$$

However, the Dirac Lagrangian Eq. (1.23) is not invariant under gauge transformations of this type since

$$
\begin{equation*}
\partial_{\mu} \psi \rightarrow e^{i \omega(x)} \partial_{\mu} \psi+i e^{i \omega(x)} \psi \partial_{\mu} \omega(x) . \tag{1.26}
\end{equation*}
$$

To built effectively a gauge field theory then, it is then necessary to couple the fermion field $\psi$ with a vector field $A^{\mu}$, which compensate for the transformations of the latter. This is made possible through the introduction of a covariant derivative:

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i q A_{\mu} \rightarrow \partial_{\mu}+i \partial_{\mu} \omega(x)-i q A_{\mu}-i \partial_{\mu} \omega(x) \tag{1.27}
\end{equation*}
$$

The Lagrangian of quantum electrodynamics (QED), then, is written in the form:

$$
\begin{equation*}
\mathscr{L}\left[\psi, \partial_{\mu} \psi, A, \partial_{\mu} A\right]=\bar{\psi}(i \not D-m) \psi-\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \tag{1.28}
\end{equation*}
$$

$$
\begin{array}{ll}
t^{1}=\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \quad t^{2}=\left[\begin{array}{ccc}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \quad t^{3}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right] \quad t^{4}=\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right] \\
t^{5}=\left[\begin{array}{ccc}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{array}\right] \quad t^{6}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right] \quad t^{7}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right] \quad t^{8}=\frac{1}{\sqrt{3}}\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right]
\end{array}
$$

Table 1.1: The 3 representation of Gell-Mann matrices, generators of $S U(3)$.

This fact tells us that we cannot realise a gauge invariant theory made of fermions alone and, instead, they need to be coupled to some sort of vector field, in this case, photons.

In the case of color charges and quantum chromodynamics (QCD), the Lagrangian is required to exhibit invariance under the more intricate $S U\left(N_{c}\right)$ gauge transformations, where $N_{c}$ represents the number of colors in the theory. This translates into invariance upon the following infinitesimal field transformations

$$
\begin{align*}
\psi_{i}(x) & \rightarrow\left(1+i \omega^{a}(x) t^{a}\right) \psi_{i}(x) \\
A^{a}{ }_{\mu} & \rightarrow A^{a}{ }_{\mu}+f^{a b c} A^{b}{ }_{\mu} \omega^{c}(x)+\frac{1}{g_{s}} \partial_{\mu} \omega^{a}(x) \tag{1.29}
\end{align*}
$$

where $t^{a}$ are $N_{c}^{2}-1$ matrices forming a basis of the algebra of the group, the index $i$ runs over the different number $N_{f}$ of quark flavours and $f_{a b c}$ are structure constants of the algebra, defined by the commutation relation:

$$
\begin{equation*}
\left[t^{a} ; t^{b}\right]=i f^{a b c} t^{c} \tag{1.30}
\end{equation*}
$$

In analogy with the QED case, we may want to define a covariant derivative invariant under Eq. (1.29):

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i g_{s} A^{a}{ }_{\mu} t^{a} . \tag{1.31}
\end{equation*}
$$

From these considerations, the Lagrangian of QCD can be written as follows

$$
\begin{equation*}
\mathscr{L}\left[\psi, \partial_{\mu} \psi, A, \partial_{\mu} A\right]=\sum_{j=1}^{N_{f}} \bar{\psi}_{j}(i \not D-m) \psi_{j}-\frac{1}{4} G^{a}{ }_{\mu \nu} G^{a}{ }^{\mu \nu}, \tag{1.32}
\end{equation*}
$$

where the antisymmetric field strength tensor $G^{a}{ }_{\mu \nu}$ has been defined as:

$$
\begin{equation*}
G^{a}{ }_{\mu \nu}=\partial_{\mu} A^{a}{ }_{\nu}-\partial_{\nu} A^{a}{ }_{\mu}+g_{s} .^{a b c} A^{b}{ }_{\mu} A^{c}{ }_{\nu} . \tag{1.33}
\end{equation*}
$$

A theory thus defined couples the $N_{f}$ quarks to a massless vector field, which, unlike the case of neutral photons, can manifest in $N_{c}^{2}-1$ distinct charge combinations. This phenomenon arises due to the nature of the matrices $t^{a}$, which play a pivotal role in the theory and are the Gell-Mann matrices in the case of $N_{c}=3$ (Tab. 1.1). These predominantly non-diagonal matrices allow for the possibility of quarks undergoing color changes during interactions, necessitating, consequently, the involvement of charged vector bosons in the theory: gluons.

### 1.4. Feynman Rules

Now, let us derive a general methodology that enables us to calculate cross-sections for strong processes. These cross-sections are not only the focal point of this work but also the primary quantities measured in the context of experiments conducted at the Large Hadron Collider (LHC).

To proceed, we require explicit formulas for the matrix elements corresponding to the processes under study. These matrix elements can be expressed in terms of Feynman diagrams, which serve as visual representations of particle interactions. By evaluating these Feynman diagrams and obtaining the corresponding matrix elements, we can then proceed to calculate the cross-sections for various strong processes.

The fermion propagator can be determined by evaluating the time-ordered Green function of the Dirac equation. In momentum space, it can be expressed as follows:

$$
\begin{equation*}
\xrightarrow{p}=\frac{i}{\not p-m+i \epsilon} \tag{1.34}
\end{equation*}
$$

In contrast to the fermion propagator, deriving the gluon propagator is considerably more complex and challenging. From path-integral methods we know that:

$$
\begin{equation*}
\text { ннннe }=\int \mathcal{D} A \exp \left[-i \frac{1}{4} G^{a}{ }_{\mu \nu} G^{a}{ }_{\mu \nu}\right] \tag{1.35}
\end{equation*}
$$

Unfortunately, this integral is badly divergent due to gauge freedom of the theory, which give rise to infinite configurations of fields that minimize the action and for which the integrand is 1. Consequently, we must exploit a trick by Faddeev and Popov [6] to somehow impose a constraint on the gauge direction, ensuring that all the equivalent configurations are counted only once.

Let $G(A)$ be some function that we wish to set equal to zero as a gauge-fixing condition, In this context, we can establish the following identity:

$$
\begin{equation*}
1=\int \mathcal{D} \alpha \delta\left(G\left(A^{\alpha}\right)\right) \operatorname{det}\left(\frac{\delta}{\delta \alpha} G\left(A^{\alpha}\right)\right) \tag{1.36}
\end{equation*}
$$

where $\delta\left(G\left(A^{\alpha}\right)\right)$ is a functional delta function and $A^{\alpha}$ is the gauge-transformed field. Consequently, Eq. (1.35) can be rewritten in the form

$$
\begin{equation*}
\int \mathcal{D} \alpha \int \mathcal{D} A e^{i \mathscr{S}_{\text {free }}[A]} \delta\left(G\left(A^{\alpha}\right)\right) \operatorname{det}\left(\frac{\delta}{\delta \alpha} G\left(A^{\alpha}\right)\right) \tag{1.37}
\end{equation*}
$$

Changing integration variables from $A$ to $A^{\alpha}$ and remembering that $\mathcal{D} A=\mathcal{D} A^{\alpha}, \mathscr{S}[A]=\mathscr{S}\left[A^{\alpha}\right]$, we can then specify a general gauge-fixing function of the type $G(A)=\partial^{\mu} A^{a}{ }_{\mu}-\omega^{a}$.

$$
\begin{equation*}
\int \mathcal{D} \alpha \int \mathcal{D} A e^{i \mathscr{S}_{\text {free }}[A]} \delta\left(\partial^{\mu} A^{a}{ }_{\mu}-\omega^{a}\right) \operatorname{det}\left(\frac{\delta}{\delta \alpha} G\left(A^{\alpha}\right)\right) . \tag{1.38}
\end{equation*}
$$

We can now integrate over all possible $\omega^{a}$ with a gaussian cut-off, obtaining the result:

$$
\begin{array}{r}
\mathcal{N}(\lambda) \int \mathcal{D} \omega \exp \left[-i \int d^{4} x \frac{\left(t^{a} \omega^{a}\right)^{2}}{2 \lambda}\right] \int \mathcal{D} \alpha \int \mathcal{D} A e^{i \mathscr{S}_{\text {free }}[A]} \delta\left(\partial^{\mu} A^{a}{ }_{\mu}-\omega^{a}\right) \operatorname{det}\left(\frac{\delta}{\delta \alpha} G\left(A^{\alpha}\right)\right) \\
\quad=\mathcal{N}(\lambda)\left(\int \mathcal{D} \alpha\right) \int \mathcal{D} A e^{i \mathscr{S}_{\text {free }}[A]} \exp \left[-i \int d^{4} x \frac{1}{2 \lambda}\left(t^{a} \partial^{\mu} A^{a}{ }_{\mu}\right)^{2}\right] \operatorname{det}\left(\frac{\delta}{\delta \alpha} G\left(A^{\alpha}\right)\right) . \tag{1.39}
\end{array}
$$

Effectively, it is like we added a fixing term $\mathscr{L}_{\text {fixing }}=-\left(t^{a} \partial^{\mu} A^{a}{ }_{\mu}\right)^{2} / 2 \lambda$ to Eq. (1.32), which allows the calculation of a gluon propagator in the form:

$$
\begin{equation*}
\underset{a, \mu}{\text { Mlllel }_{b, \nu}}=\delta^{a b} \frac{i}{p^{2}}\left(-g_{\mu \nu}+(1-\lambda) \frac{p_{\mu} p_{\nu}}{p^{2}}\right) \tag{1.40}
\end{equation*}
$$

This covariant class of gauge conditions leads to a relatively straightforward expression for the propagator. Notably, the case with $\lambda=1$, commonly referred to as the Feynman-'t Hooft gauge, is particularly convenient and widely used in the literature.

With this Faddeev-Popov procedure exploited, however, another extra term depending from $A$ arises:

$$
\begin{equation*}
\operatorname{det}\left(\frac{\delta}{\delta \alpha} G\left(A^{\alpha}\right)\right)=\operatorname{det}\left(\frac{1}{g_{s}} \partial^{\mu} D_{\mu}\right) \tag{1.41}
\end{equation*}
$$

From wisdom in Grassmann-Berezin calculus ${ }^{4}$, we can express this determinant in terms of a further path integral

$$
\begin{equation*}
\operatorname{det}\left(\frac{1}{g_{s}} \partial^{\mu} D_{\mu}\right)=\int \mathcal{D} \eta \mathcal{D} \bar{\eta} \exp \left[i \int d^{4} x \bar{\eta}\left(-\partial^{\mu} D_{\mu}\right) \eta\right], \tag{1.42}
\end{equation*}
$$

which eventually adds another extra-term to the Lagrangian Eq. (1.32):

$$
\begin{equation*}
\mathscr{L}_{\text {ghost }}=\bar{\eta}^{a}\left(-\partial^{2} \delta^{a c}-g_{s} \partial^{\mu} f^{a b c} A^{b}{ }_{\mu}\right) \eta^{c} . \tag{1.43}
\end{equation*}
$$

The introduction of these new fields, described by Grassmann numbers, gives rise to pseudoparticles known as ghosts in the theory. The first term in the previous equation allows us to write a ghost propagator in terms of:

$$
\begin{equation*}
{ }_{a}{ }^{p} \ldots \ldots \ldots \ldots \ldots \ldots=\frac{i \delta^{a c}}{p^{2}+i \epsilon} \tag{1.44}
\end{equation*}
$$

It is important to emphasize that these ghost particles are gauge-dependent entities. In fact, if we had chosen a different gauge-fixing term, such as $\mathscr{L}_{\text {fixing }}=-\left(t^{a} n^{\mu} A^{a} \mu\right)^{2} / 2 \lambda$, it would have resulted in the absence of ghosts in the theory. However, in this axial gauge, the gluon propagator becomes significantly more complicated:

$$
\begin{equation*}
{\left.\underset{a, \mu}{p} \text { uneue }_{b, \nu}=\delta^{a b} \frac{i}{p^{2}}\left(-g_{\mu \nu}+\frac{n_{\mu} p_{\nu}+n_{\nu} p_{\mu}}{(n \cdot p)}-\frac{\left(n^{2}+\lambda p^{2}\right) p_{\mu} p_{\nu}}{(n \cdot p)^{2}}\right)\right) ~}_{(n \cdot)^{2}} \tag{1.45}
\end{equation*}
$$

but can be simplified by a proper tuning of the parameters $n^{2}$ and $\lambda$. A common practice in the literature is to set these parameters to zero, resulting in a simplified form of the gluon propagator known as the light-cone gauge.

[^3]where $x_{i}$ is a generic complex number. Integrals over Grassmann numbers are defined through Berezin integration, which follows the fundamental rules:
$$
\int d \eta 1=\frac{d}{d \eta} 1=0 \quad \int d \eta \eta=\frac{d}{d \eta} \eta=1
$$
and is surprisingly defined as the same of derivation. Additionally, a complex conjugation $\bar{\eta}$ can be introduced to satisfy the condition:
$$
\int d \bar{\eta} d \eta(\eta \bar{\eta})=1
$$

From a combinations of these arguments, derivation of Eq. (1.42) is immediate. For further information we refer the interested reader to ref. [7].


$$
\begin{array}{r}
-g f^{a b c}\left[\left(p_{1}-p_{2}\right)^{\gamma} g^{\alpha \beta}\right. \\
+\left(p_{2}-p_{3}\right)^{\alpha} g^{\beta \gamma} \\
\left.+\left(p_{3}-p_{1}\right)^{\beta} g^{\gamma \alpha}\right]
\end{array}
$$



$$
\begin{array}{r}
-i g^{2}\left[f^{e a c} f^{e b d}\left(g^{\alpha \beta} g^{\gamma \delta}-g^{\alpha \delta} g^{\beta \gamma}\right)\right. \\
\quad+f^{e a d} f^{e b c}\left(g^{\alpha \beta} g^{\gamma \delta}-g^{\alpha \gamma} g^{\beta \delta}\right) \\
\left.+f^{e a b} f^{a c d}\left(g^{\alpha \gamma} g^{\beta \delta}-g^{\alpha \delta} g^{\beta \gamma}\right)\right]
\end{array}
$$

Table 1.2: QCD Feynman rules in the Feynman gauge

### 1.5. Renormalization and Asymptotic freedom

Rules derived in the previous section and enlisted in Tab. 1.2 describe the dynamics of free quarks, gluons and ghosts. However, being QCD a theory of interactions, we are interested in the derivation of exact propagators that take in account radiative corrections where quarks are able to absorb and emit gluons (Fig. 1.2).


Figure 1.2: Quark-gluon scattering: LO and NLO radiative corrections
It is worth noting that these radiative corrections solely consist of adding extra-gluon lines in Feynman diagrams, with each line contributing a multiplicative factor proportional to the coupling constant $\alpha_{s}=g_{s}^{2} / 4 \pi$. As a result, general observables $\sigma$ can be expanded as perturbative series in $\alpha_{s}$, whose convergence, however, is not guaranteed and can be assured only for $\alpha_{s}<1$.

In general, any dimensionless $\sigma$ thus expanded depends on a set of dimensionless parameters $\{y\}$ and on a certain energy hard scale $Q$ characteristic of the process. For high values of $Q$, we can neglect all the other dimensional parameters of the theory (e.g. the masses of the particles), with the dimensional analysis now implying that $\sigma$ should be independent of Q .

We can then renormalize the theory to get rid of the UV divergences in perturbative calculations by introducing a new mass scale $\mu_{R}$ (renormalization scale) and rewriting $\sigma$ as a now convergent series in a new $\alpha_{s}\left(Q^{2}\right)$, instead of the old $\alpha_{s}$. Being $\mu_{R}$ an arbitrary parameter, physical observables such as $\sigma$ should not acquire new extra dependencies on it after the renormalization. This statement equals to fulfilling the following Callan-Symanzik or Renormalization Group equation (RGE):

$$
\begin{equation*}
\mu_{R}^{2} \frac{d}{d \mu_{R}^{2}} \sigma\left(\frac{Q^{2}}{\mu_{R}^{2}}, \alpha_{s},\{y\}\right)=\left[\mu_{R}^{2} \frac{\partial}{\partial \mu_{R}^{2}}+\beta\left(\alpha_{s}\right) \frac{\partial}{\partial \alpha_{s}}\right] \sigma\left(\frac{Q^{2}}{\mu_{R}^{2}}, \alpha_{s},\{y\}\right)=0 \tag{1.46}
\end{equation*}
$$

where we have defined the beta function

$$
\begin{equation*}
\beta\left(\alpha_{s}\right)=\mu_{R}^{2} \frac{\partial \alpha_{s}}{\partial \mu_{R}^{2}} . \tag{1.47}
\end{equation*}
$$

The most general expression for a renormalized $\sigma$ will then be given by $\sigma\left(1, \alpha_{s}\left(Q^{2}\right),\{y\}\right)$ with the running coupling constant $\alpha_{s}\left(Q^{2}\right)$ solving the Cauchy Problem

$$
\begin{equation*}
\mu_{R}^{2} \frac{\partial \alpha_{s}\left(\mu_{R}^{2}\right)}{\partial \mu_{R}^{2}}=\beta\left(\alpha_{s}\right), \quad \alpha_{s}\left(\mu_{R}\right)=\alpha_{s} \tag{1.48}
\end{equation*}
$$

and where the $\beta\left(\alpha_{s}\right)$ can also be expanded in $\alpha_{s}$ as

$$
\begin{equation*}
\beta\left(\alpha_{s}\right)=-\beta_{0} \alpha_{s}^{2}+O\left(\alpha_{s}^{3}\right) \quad \text { with } \quad \beta_{0}=\frac{11 N_{c}-2 N_{f}}{12 \pi}>0 \tag{1.49}
\end{equation*}
$$

leading to a solution

$$
\begin{equation*}
\alpha_{s}\left(Q^{2}\right)=\frac{\alpha_{s}\left(\mu_{R}^{2}\right)}{1+\alpha_{s}\left(\mu_{R}^{2}\right) \beta_{0} \ln \frac{Q^{2}}{\mu_{R}^{2}}} . \tag{1.50}
\end{equation*}
$$



Figure 1.3: A multi-jet event recorded by the CMS detector at the Large Hadron Collider

With this definition, the QCD running coupling constant decreases as the energy increase and this fact, known under the name of asymptotic freedom, allows perturbative analysis at least in the limit of high energies.

Nevertheless, at low energies this is not possible and the expressions need to be resummed at all orders in $\alpha_{s}$. In this regime QCD is observed to behave in a rather odd way not totally understood nowadays where all of the asymptotic states of the theory at finite energies transform trivially under $S U(3)$ (i.e. are colourless), thus implying that coloured states like quarks and gluons can only exist as constituents of larger hadrons.

This crucial feature of colour confinement is responsible for one of the most spectacular events in particle physics: the production of hadron jets in hadron-hadron collisions (Fig. 1.3 and Cover). During these collisions, colourless hadrons are shattered into fragments, with each fragment carrying away a portion of the color charge. However, in order to obey confinement, these fragments have to create additional colored objects around them in an effort to form colorless objects, starting chain reaction where the produced hadrons tend to travel all in the same direction, forming an almost collimated jet.

### 1.6. The Parton Model: Collinear and Soft Singularities

Therefore now, the crucial task is to bridge the calculations obtained in terms of the fundamental constituents of the theory (quark and gluons) with the measured quantities we are able to observe in the asymptotic states (hadrons).

This can be attained under the assumptions of the Parton Model [8, 9], where hadrons are considered as a loosely bound assemblage of a small number of constituents, called partons, which we identify with quarks and gluons.

We can observe that short-distances interactions, which involve short time-scales, can be
easily related to high energy ones accordingly to the uncertainty principle. Thus, asymptotic freedom holds and quarks and gluons can be considered as freely moving in the hadron, carrying a certain fraction $x$ of its total momentum. The collision between two hadrons $h_{1}$ and $h_{2}$, then, can be summarized with the collision of pair of partons with defined momentum, where contribution from multiple scattering are typically considered negligible. We can express the hadronic cross-section, denoted as $\sigma$, in terms of partonic cross-sections $\hat{\sigma}$ :

$$
\begin{equation*}
\sigma_{h_{1} h_{2}}\left(\tau, Q^{2}\right)=\sum_{i \in h_{1}, j \in h_{2}} \int_{\tau}^{1} d x_{1} f_{i}\left(x_{1}\right) \int_{\frac{\tau}{x_{1}}}^{1} d x_{2} f_{j}\left(x_{2}\right) \hat{\sigma}_{i j}\left(\frac{\tau}{x_{1} x_{2}}, Q^{2}\right) \tag{1.51}
\end{equation*}
$$

where the functions $f_{i}(x)$ are parton distribution functions (PDF), which quantify the probability of finding a specific parton $i$ inside the hadron, carrying a fraction $x$ of the hadron's total momentum.

Using the Parton Model, so, we can define general factorised expressions for a general hadronic observable $\mathcal{O}$ in terms of a parton luminosity, defined as the convolution of all the PDFs

$$
\begin{equation*}
\mathcal{O}=\mathcal{L} \otimes \hat{\mathcal{O}} \tag{1.52}
\end{equation*}
$$

where $\otimes$ stands for the multiplicative convolution, and $\hat{\mathcal{O}}$ is the partonic observable which we saw can be expanded in terms of Feynman diagrams.

This new recipe, in addition to allowing for the resolution of the problem stated at the beginning of this section, will be crucial for the discussion of the statements that will follow. In perturbative calculations, UV divergences are not the only to appear, in fact, when we consider diagrams as the ones in Fig. 1.2, we can observe that the radiative corrections bring a contribution which is divergent when the extra-fermion propagator goes on-shell. If we try to compute the matrix element associated with the diagram (B), for instance:

$$
\begin{align*}
& \text { |cc|cc|}  \tag{1.53}\\
& \propto \int \frac{d^{3} k}{(2 \pi)^{3}}|\mathcal{M}(p, k \rightarrow p-k)|^{2}\left|\frac{i}{\not p-\not k-m}\right|^{2}\left|\hat{\sigma}^{(0)}\left(q, p-k \rightarrow p^{\prime}\right)\right| \\
& =\int \frac{d k_{t}^{2} d z}{2(1-z)(2 \pi)^{3}} \frac{1}{k_{t}^{2} /(1-z)}|\mathcal{M}(p, k \rightarrow z p)|^{2} \hat{\sigma}^{(0)}\left(q, z p \rightarrow p^{\prime}\right)
\end{align*}
$$

where we used Sudakov parametrization $k=(1-z) p+k_{t}+\eta$ such that $p^{2}=k^{2}=\eta^{2}=p \cdot k_{t}=$ $\eta \cdot k_{t}=0$. This expression then shows a logarithmic divergence when $k_{t} \rightarrow 0$, i.e. when the gluon is emitted in the same direction of the propagating quark. If we introduce a cut-off $\mu_{F}$ to tame this collinear singularity, in analogy with the renormalization techniques of the previous section, and integrate over $k_{t}^{2}$, we obtain something in the form:

$$
\begin{equation*}
\hat{\sigma}=\hat{\sigma}^{(0)}(p)+\frac{\alpha_{s}}{2 \pi} \ln \frac{Q^{2}}{\mu_{F}^{2}} \int d z P(z) \hat{\sigma}^{(0)}(z p) \tag{1.54}
\end{equation*}
$$

The strenght of this argument lies in its universality: since we never used either an explicit form of $\sigma^{(0)}$ or for the radiative correction vertex $\mathcal{M}(p, k \rightarrow z p)$, we can write the correction to
a general partonic observable due to collinear emission in terms of a multiplicative convolution:

$$
\begin{equation*}
\hat{\sigma}_{i}=\hat{\sigma}_{i}^{(0)}(p)+\frac{\alpha_{s}}{2 \pi} \ln \frac{Q^{2}}{\mu_{F}^{2}} \int d z P_{i j}(z) \hat{\sigma}_{j}^{(0)}(z p) \tag{1.55}
\end{equation*}
$$

These splitting functions $P_{i j}(z)$ are defined to represent the probability of a parton $i$ splitting into another parton of flavor $j$ carrying a fraction $z$ of its total momentum. By their definition, splitting functions and PDFs are connected through a set of RG equations

$$
\begin{equation*}
Q^{2} \frac{d}{d Q^{2}} f_{i}\left(x, Q^{2}\right)=\sum_{j} P_{i j}\left(\alpha_{s}\left(Q^{2}\right), x\right) \otimes f_{j}\left(Q^{2}\right)\left(x, Q^{2}\right) \tag{1.56}
\end{equation*}
$$

known in literature as Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations (DGLAP) [1012]. In a process similar to renormalization, splitting functions and PDFs can be redesigned through DGLAP equations in way that divergences associated with collinear singularities can be effectively regulated and removed, resulting in well-behaved and physically meaningful results.

In the previous section we wrote a particular RGE Eq. (1.46) for physical observables. However, with the introduction of the Parton Model, if this equation continues to hold for hadronic cross-section, the same can't be said for partonic ones, that, in exchange, ceased to be physical observables. In the conjugate space ${ }^{5}$, it is possible to derive a new RGE equation for partonic observables starting from Eq. (1.46):

$$
\begin{gather*}
{\left[\mu_{R}^{2} \frac{\partial}{\partial \mu_{R}^{2}}+\beta\left(\alpha_{s}\right) \frac{\partial}{\partial \alpha_{s}}\right]\left(A_{N}\left(\mu_{R}^{2}\right) C_{N}\left(\frac{Q^{2}}{\mu_{R}^{2}}, \alpha_{s}\left(\mu_{R}^{2}\right)\right)\right)=0} \\
\text { with } \quad \mu_{R}^{2} \frac{d}{d \mu_{R}^{2}} A_{N}\left(\mu_{R}^{2}\right)=\gamma_{N}\left(\mu_{R}^{2}\right) A_{N}\left(\mu_{R}^{2}\right)  \tag{1.57}\\
\Longrightarrow\left[\mu_{R}^{2} \frac{\partial}{\partial \mu_{R}^{2}}+\beta\left(\alpha_{s}\right) \frac{\partial}{\partial \alpha_{s}}+\gamma_{N}\left(\alpha_{s}\left(\mu_{R}^{2}\right)\right)\right] C_{N}\left(\frac{Q^{2}}{\mu_{R}^{2}}, \alpha_{s}\left(\mu_{R}^{2}\right)\right)=0
\end{gather*}
$$

where we have included an anomalous dimension $\gamma_{N}$ that takes in account for the evolution of the PDFs given by DGLAP. The general solution to this equation will then provide a running matrix element in the form

$$
\begin{equation*}
C_{N}\left(\frac{Q^{2}}{\mu_{R}^{2}}, \alpha_{s}\left(\mu_{R}^{2}\right)\right) A_{N}\left(\mu_{R}^{2}\right)=C_{N}\left(1, \alpha_{s}\left(Q^{2}\right)\right)\left[\exp \int_{\mu_{R}^{2}}^{Q^{2}} \frac{d \lambda^{2}}{\lambda^{2}} \gamma_{N}\left(\alpha_{s}\left(\lambda^{2}\right)\right)\right] A_{N}\left(\mu_{R}^{2}\right) \tag{1.58}
\end{equation*}
$$

where we can expand $\gamma_{N}\left(\alpha_{s}\right)=\gamma_{N}^{(0)}\left(\alpha_{s}\right)+O\left(\alpha_{s}^{2}\right)$, in a way analogous to Eq. (1.49) and obtain, up to NLO precision:

$$
\begin{gather*}
C_{N}\left(\alpha_{s}\left(\mu_{R}^{2}\right)\right) A_{N}=\left(C_{N}^{(0)}+\alpha_{s}\left(Q^{2}\right) C_{N}^{(1)}\right)\left[1+\alpha_{s}\left(\mu_{R}^{2}\right) \gamma_{N}^{(0)} \ln \frac{Q^{2}}{\mu_{R}^{2}}+O\left(\alpha_{s}^{2}\left(\mu_{R}^{2}\right)\right)\right] A_{N}\left(\mu_{R}^{2}\right) \\
\rightarrow \hat{\sigma}=\left[\hat{\sigma}^{(0)}+\alpha_{s}\left(\mu_{R}^{2}\right) \ln \frac{Q^{2}}{\mu_{R}^{2}} P^{(0)} \otimes \hat{\sigma}^{(0)}+\alpha_{s}\left(Q^{2}\right) \hat{\sigma}^{(1)}\right] \otimes \mathcal{L}\left(\mu_{R}^{2}\right)+O\left(\alpha_{s}^{2}\right) \tag{1.59}
\end{gather*}
$$

As expected, this equation exhibits an UV logarithm coming from the renormalization process which is completely analogous to the collinear one in Eq. (1.55) obtained with the explicit calculation of Feynman diagrams. UV and collinear logarithms, in fact, can be regarded as the same object depending on the framework we adopt: they either can be the result of an UV

[^4]energy cut-off on matrix elements to be reabsorbed by the definition of a running coupling constant; or the consequence of an IR cut-off on transverse momentum values which can then be resummed in the calculation of running splitting functions with DGLAP.

Another subtlety, however, has to be considered. Collinear singularities actually include, by their definition, all the soft singularities corresponding to the emission of gluons with total momentum close to zero $(z \rightarrow 1$, somehow a consequence of the uncertainty principle, how can we distinguish the presence of a massless gluon at rest?). This is due to the fact that, in the same way we could wrote a universal logarithmic factor in Eq. (1.55) with Sudakov parametrization, the on shell fermion propagator in the $k$-space adds a universal eikonal factor:

$$
\begin{equation*}
\frac{i}{\not p-\not k-m} \propto \frac{p^{\mu}}{p \cdot k} \tag{1.60}
\end{equation*}
$$

divergent in the limit $k \rightarrow 0$. Further confirmation of the existence of soft singularities can be found in the very expression of the quark-quark splitting function

$$
\begin{equation*}
P_{q q}(z)=C_{F}\left[\frac{1+z^{2}}{1-z}+\frac{3}{2} \delta(1-z)\right] \tag{1.61}
\end{equation*}
$$

that, when integrated, displays the presence of an extra soft logarithm due to the factor $(1-z)_{+}^{-1}$.

Nonetheless, while the presence of IR singularities in individual terms of calculations can be mathematically problematic, the validity of Kinoshita-Lee-Nauenberg theorem [13, 14] guarantees that these singularities are absorbed and canceled out by the appropriate combination of real and virtual contributions as a consequence of the structure of the gluon emission and absorption processes within the perturbative framework. Soft gluon emissions (such as Fig. 1.2 (B) and (C)) contribute to real emissions, while virtual loop diagrams (D) account for the absorption of these soft gluons by other particles in the process.

It's then crucial, in order to obtain sensible predictions in the low energy regime, to take in account for all the orders of logarithms coming from the removed singularities. For this reason, in the next section, we will examine resummation techniques that specifically address the treatment of logarithms arising from both soft and collinear emissions.


## Threshold and Transverse Momentum Resummations

In this chapter we will perform resummations of the soft and collinear logarithms arising in the case of a process where two hadrons $h_{1}$ and $h_{2}$ collide to form a final state $H$ and radiation $X$

$$
\begin{equation*}
h_{1}\left(p_{1}\right)+h_{2}\left(p_{2}\right) \rightarrow H\left(p_{H}\right)+X\left(k_{1}, \cdots k_{k+1}\right) \tag{2.1}
\end{equation*}
$$

particularly focusing on the calculation of transverse momentum distributions $d \sigma / d p_{T}^{2}$, where $p_{T}$ is the transverse momentum of $H$ with respect to the direction of the colliding hadrons. This particular scenario holds significant relevance in modern particle physics, as it encompasses various processes, including the production of the Higgs boson through gluon fusion. Such processes serve as critical benchmarks for potential evidence of physics beyond the Standard Model (BSM) and are therefore of great interest and importance in experimental studies [15].

Transverse momentum distributions can be computed through the formula:

$$
\begin{equation*}
\frac{d \sigma}{d p_{T}}=|\mathcal{M}|^{2} \frac{d \Phi}{d p_{T}} \tag{2.2}
\end{equation*}
$$

where $\mathcal{M}$ stands for the exact matrix element representing the process, to be computed perturbatively from diagrams, and $\Phi$ is the available phase space. As stated at the end of the previous chapter, in order to effectively estimate transverse momentum distributions it will be necessary to resum all orders of logarithms. In Sec. 2.2 and 2.3 we will expose a general technique to perform threshold resummations [16], i.e. of soft logarithms, as in the general formulation provided by ref. [17]. Then in Sec. 2.4 and 2.5 , we will face the issues related to transverse momentum resummations [18], i.e. of collinear logarithms, in the general formulation of refs. [19, 20].

### 2.1. Kinematic identities and notation

First, it is important to underline the presence of two important scales characterising the process: an hard scale given by the mass $M$ of the state $H$, and a soft one provided by its transverse momentum $p_{T}$. This choice of scales, however, is not unique and, in fact, it will be extremely convenient in the sequel to exploit other choices of hard scales, such as the ones defined by:

$$
\begin{equation*}
Q \equiv \sqrt{M^{2}+p_{T}^{2}}+\sqrt{p_{T}^{2}} \quad \text { or } \quad Q p_{T}=\frac{Q^{2}-M^{2}}{2} \tag{2.3}
\end{equation*}
$$

where we should note that since the hard scale thus defined are all independent, any choice among $\left\{Q^{2}, M^{2}, Q p_{T}\right\}$ is completely equivalent.

Moreover, it will be important to define two dimensionless scaling variables which will allow the factorisation of the observables under suitable integral transformations:

$$
\begin{equation*}
x \equiv \frac{Q^{2}}{\hat{s}}, \quad \xi_{p} \equiv \frac{p_{T}^{2}}{Q^{2}} \quad \text { with } \quad \tau \equiv \frac{Q^{2}}{s} \tag{2.4}
\end{equation*}
$$

where $s$ and $\hat{s}=x_{1} x_{2} s$ represent the hadron and partonic centre of mass energies respectively. We can observe that $x$ represents the minimum fraction of $\hat{s}$ employed for the formation of an $H$ with fixed $p_{T}$, so $0<x<1$ and soft emissions are coherently described by the limit $x \rightarrow 1$. Furthermore, is then possible to write an equivalent of Eq. (1.51) for transverse momentum distributions as:

$$
\begin{equation*}
\frac{d \sigma}{d \xi_{p}}\left(\tau, \xi_{p}\right)=\tau \sum_{i, j} \int_{\tau}^{1} \frac{d x}{x} \mathcal{L}_{i j}\left(\frac{\tau}{x}\right) \frac{1}{x} \frac{d \hat{\sigma}_{i j}}{d \xi_{p}}\left(\tau, \xi_{p}\right) \tag{2.5}
\end{equation*}
$$

which factorises in conjugate space by means of Sec. A.1:

$$
\begin{equation*}
\frac{d \sigma}{d \xi_{p}}\left(N, \xi_{p}\right)=\sum_{i, j} \mathcal{L}_{i j}\left(N+1, \xi_{p}\right) \frac{d \hat{\sigma}_{i j}}{d \xi_{p}}\left(N, \xi_{p}\right) \tag{2.6}
\end{equation*}
$$

When doing explicit calculations, resummation is most easily expressed if we define a coefficient function $C\left(N, \xi_{p}\right)$ that factors out the Born-level expression. In the conjugate space approach we are going to follow, we will divide this coefficient into a process depend hard part $H\left(N, \xi_{p}\right)$ and one (or more) universal "jet" functions $J\left(N, \xi_{p}\right)$ which enclose all the multiplicative factors brought by the radiative corrections, including the divergences. This will be achieved through the requirement of further factorization properties on phase space, at least in certain particular kinematic limits, leading to an expression of transverse momentum distributions in terms of

$$
\begin{equation*}
\frac{d \hat{\sigma}}{d \xi_{p}}\left(N, \xi_{p}\right)=C\left(N, \xi_{p}\right){\frac{d \hat{\sigma}}{d \xi_{p}}}^{(0)}\left(N, \xi_{p}\right)=H\left(N, \xi_{p}\right) J\left(N, \xi_{p}\right){\frac{d \hat{\sigma}^{(0)}}{d \xi_{p}}}^{\left(N, \xi_{p}\right)} \tag{2.7}
\end{equation*}
$$

### 2.2. Phase space factorisation at Threshold

We will then start with the resummation of soft logarithms when the value of $p_{T}$ is fixed and non zero. In order to perform this effectively, we may want to complete a factorisation of the phase space in the particular kinematic limit of $x \rightarrow 1$ at fixed $p_{T}$.

At threshold, the centre of mass energy of the system approaches its minimum with the consequence that the energy is only enough to produce the final state $X$, while the invariant mass $W$ of the extra-radiation approaches zero, with the implication:

$$
\begin{equation*}
W=\sum_{i>j}^{n}\left|k_{i}\right|\left|k_{j}\right|\left(1-\cos \theta_{i j}\right) \rightarrow 0 \tag{2.8}
\end{equation*}
$$

where $\left|k_{i}\right|$ stands for the modulus of the three-momentum and $\theta_{i j}$ the angle enclosed between the directions of the momenta $k_{i}$ and $k_{j}$.

For the consistency of a $p_{T}$ distribution, there must be at least one non-soft parton in $X$ that recoils against $H$ forcing him into the direction implied by its particular value of $p_{T}$. Hence, at threshold, the extra-emitted partons have either to be all soft or collinear to this non-soft one. We can then assume in the radiation $X$ the existence of $n<k$ soft $k_{i}$, and rename all the
$m+1=k+1-n$ momenta of non-soft partons as $k_{j}^{\prime}$, which turn into a cinematic configuration of soft limit given by the conditions:

$$
\begin{array}{ll}
k_{i}=0 & 1 \leq i \leq n \\
\theta_{i j}=0 & \sum_{j=1}^{m+1}{k_{j}^{\prime 0}}^{0}=p_{H} \quad 1 \leq i, j \leq m+1 \tag{2.9}
\end{array}
$$

Under these assumptions, the phase space measure can be rewritten as

$$
\begin{gather*}
d \phi_{m+n+2}\left(p_{1}, p_{2} ; p_{H}, k_{1}, \cdots k_{n}, k_{1}^{\prime}, \cdots k_{m+1}^{\prime}\right)= \\
=\int \frac{d q^{2}}{2 \pi} \int \frac{d k^{\prime 2}}{2 \pi} d \phi_{n+1}\left(p_{1}, p_{2} ; q, k_{1}, \cdots k_{n}\right) d \phi_{2}\left(q ; p_{H}, k^{\prime}\right) d \phi_{m+1}\left(k^{\prime} ; k_{1}^{\prime}, \cdots k_{m+1}^{\prime}\right) \tag{2.10}
\end{gather*}
$$

where we factorised:

- an inclusive phase space measure $d \phi_{n+1}$ for the production of massive object of mass $q^{2}$ and an extra-radiation of $n$ soft partons $k_{i}$, starting from two incoming partons with momenta $p_{1}, p_{2}$. This is the same as a DY or Higgs production process, which we know that, in the soft limit, can be written in terms of a dimensionless integration measure times a dimensional factor in powers of a scale:

$$
\begin{equation*}
\Lambda_{\mathrm{DY}}^{2} \equiv Q^{2}(1-x)^{2} \tag{2.11}
\end{equation*}
$$

as stated in Eq. (4.30) in [21].

- the phase space measure $d \phi_{2}$ for the division of the $q^{2}$-massive object into a final state with mass $M$, momentum $p_{H}$ and fixed transverse momentum $p_{T}$, and another object with momentum $k^{\prime}$ recoiling against it. In $d=4-2 \epsilon^{1}$, this term can be written as Eq. (2.23) of [17]:

$$
\begin{gather*}
d \phi_{2}\left(q ; p_{H}, k^{\prime}\right)=\frac{d^{d-1} k^{\prime}}{(2 \pi)^{d-1} 2 k^{\prime 0}} \frac{d^{d-1} p_{H}}{(2 \pi)^{d-1} 2 p_{H}^{0}}(2 \pi)^{d} \delta^{(d)}\left(q-k^{\prime}-p_{H}\right)  \tag{2.12}\\
=\frac{(4 \pi)^{\epsilon}}{16 \pi \Gamma(1-\epsilon)} \frac{p_{T}^{-2 \epsilon}}{k_{0}^{\prime} p_{H}^{0}} d p_{T}^{2} \delta\left(p_{H}^{0}+k_{0}^{\prime}-\sqrt{q^{2}}\right)
\end{gather*}
$$

- a phase space measure $d \phi_{m+1}$ for the production, starting an object of momentum $k^{\prime}$, of a final system of $m+1$ collinear partons with momenta $k_{j}^{\prime}$, analogous to a DIS. Accordingly to Eq. (4.17) of [21], this too can be written in terms of a dimensionless integration measure, with a dimensional factor in terms of powers of

$$
\begin{equation*}
\Lambda_{\mathrm{DIS}}^{2} \equiv Q p_{T}(1-x) \tag{2.13}
\end{equation*}
$$

If we define two dimensionless coefficients $v=\left(p_{H}^{0} / q\right)^{2}$ and $1-w=\left(k_{0}^{\prime} / p_{H}^{0}\right)^{2}$, this particular decomposition of the phase space joint with the momentum conservation condition turns into:

$$
\begin{equation*}
d \phi_{m+n+2} \equiv d \Phi\left(\Lambda_{\mathrm{DY}}^{2}\right) d \Psi\left(\Lambda_{\mathrm{DIS}}^{2}\right) \delta(x-v w) \tag{2.14}
\end{equation*}
$$

for an appropriate definition of the measures $d \Phi, d \Psi$. The integration over a phase space thus written allows for an expression of the transverse momentum distribution in terms of multiplicative convolution, which factorizes in Mellin space.

[^5]

Figure 2.1: Diagrammatic representation of the factorisation of Eq. (2.10)

### 2.3. Resummation at Threshold

This result suggests that a proper threshold resummation would require the exploitation of renormalization techniques acting on two different scales, with a factorization of the coefficient function of Eq. (2.7) in terms of:

$$
\begin{equation*}
\ln C\left(N, Q^{2}\right)=\ln H\left(Q^{2}\right)+\ln J_{a_{1}}\left(\Lambda_{a_{1}}^{2}\right)+\ln J_{a_{1}}\left(\Lambda_{a_{1}}^{2}\right) \tag{2.15}
\end{equation*}
$$

where we distinguished two different jet functions for the contributions deriving from $a_{1}=\mathrm{DY}$ and $a_{2}=$ DIS contributions. Since $C$ is not a physical observable, it is then possible to define a physical anomalous dimension:

$$
\begin{align*}
& \gamma^{\text {phys }}=\mu^{2} \frac{d}{d \mu^{2}}\left(\ln H\left(\frac{Q^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)+\ln J_{a_{1}}\left(\left(\frac{\Lambda_{a_{1}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)\right)+\ln J_{a_{2}}\left(\frac{\Lambda_{a_{2}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)\right) \\
& \equiv \gamma^{c}\left(\frac{Q^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)+\gamma^{l_{1}}\left(\frac{\Lambda_{a_{1}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)+\gamma^{l_{2}}\left(\frac{\Lambda_{a_{2}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right) \\
& \text { with } \quad \gamma^{c}\left(\frac{Q^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)=\mu^{2} \frac{d}{d \mu^{2}} \ln H, \quad \gamma^{l_{i}}\left(\frac{\Lambda_{a_{i}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)=\mu^{2} \frac{d}{d \mu^{2}} \ln J_{a_{i}} \tag{2.16}
\end{align*}
$$

We should then note that while $\gamma^{\text {phys }}$ is renomalization-group invariant, ts components $\gamma^{c}, \gamma^{l_{1}}, \gamma^{l_{2}}$ are not. Nevertheless, they can be related by the equation:

$$
\begin{align*}
& \mu^{2} \frac{d}{d \mu^{2}}\left(\gamma^{c}\left(\frac{Q^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)\right)=-\mu^{2} \frac{d}{d \mu^{2}}\left(\gamma^{l_{1}}\left(\frac{\Lambda_{a_{1}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)\right)-\mu^{2} \frac{d}{d \mu^{2}}\left(\gamma^{l_{2}}\left(\frac{\Lambda_{a_{2}}^{2}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)\right) \\
& \equiv \bar{g}_{1}\left(\alpha_{s}\left(\mu^{2}\right)\right)+\bar{g}_{2}\left(\alpha_{s}\left(\mu^{2}\right)\right) \tag{2.17}
\end{align*}
$$

which provides a useful expression for $\gamma^{\text {phys }}$

$$
\begin{equation*}
\gamma^{\mathrm{phys}}=\bar{g}_{0}\left(\alpha_{s}\left(Q^{2}\right)\right)+\int_{Q^{2}}^{\Lambda_{a_{1}}^{2}} \frac{d \mu^{2}}{\mu^{2}} \bar{g}_{1}\left(\alpha_{s}\left(\mu^{2}\right)\right)+\int_{Q^{2}}^{\Lambda_{a_{2}}^{2}} \frac{d \mu^{2}}{\mu^{2}} \bar{g}_{2}\left(\alpha_{s}\left(\mu^{2}\right)\right) \tag{2.18}
\end{equation*}
$$

In Mellin space, the dependence on the scale $\Lambda$ is expressed in terms of certain $\bar{\Lambda}$, having:

$$
\begin{equation*}
\Lambda_{a}^{2}\left(z, \lambda_{a}^{2}\right)=Q^{2}(1-z)^{a} \rightarrow \bar{\Lambda}_{a}^{2}\left(N, \lambda_{a}^{2}\right)=\frac{\lambda_{a}^{2}}{N^{a}} \tag{2.19}
\end{equation*}
$$

with $\lambda_{a}^{2}=Q^{2}$ for DY, $\lambda_{a}^{2}=Q p_{T}$ for DIS. Allowing for an expression of the coefficient function where we resummed the radiative contributions in terms of:

$$
\begin{gather*}
C_{i j}^{(\mathrm{res})}\left(N, \frac{Q^{2}}{\mu^{2}}, \frac{Q p_{T}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)=g_{0}^{i j}\left(\frac{Q^{2}}{\mu^{2}}, \alpha_{s}\left(Q^{2}\right)\right) \\
\times \exp \left\{\int_{1}^{N^{2}} \frac{d n}{n} \int_{n \mu^{2}}^{Q^{2}} \frac{d q^{2}}{q^{2}} \bar{g}_{1}^{(i)}\left(\alpha_{s}\left(q^{2} / n\right)\right)+\int_{1}^{N} \frac{d n}{n} \int_{n \mu^{2}}^{Q p_{T}} \frac{d q^{2}}{q^{2}} \bar{g}_{2}^{(j)}\left(\alpha_{s}\left(q^{2} / n\right)\right)\right\}  \tag{2.20}\\
\text { with } g_{0}^{i j}=H_{i j} \exp \bar{g}_{0}
\end{gather*}
$$

The functions $\bar{g}_{1}, \bar{g}_{2}$ can be conveniently expressed in powers of $\alpha_{s}$, being derived from explicit calculations of Feynman diagrams. With these expansions, the resummed coefficient assumes the form:

$$
\begin{align*}
C^{(\text {res })}=g_{0} & \exp \left[\ln N^{2} g_{1,1}\left(\alpha_{s} \ln N^{2}\right)+g_{1,2}\left(\alpha_{s} \ln N^{2}\right)+\alpha_{s} g_{1,3}\left(\alpha_{s} \ln N^{2}\right)+\ldots\right. \\
& \left.+\ln N g_{2,1}\left(\alpha_{s} \ln N\right)+g_{2,2}\left(\alpha_{s} \ln N\right)+\alpha_{s} g_{2,3}\left(\alpha_{s} \ln N\right)+\ldots\right] \tag{2.21}
\end{align*}
$$

Which allows easy fixed logarithmic order calculations as the ones that will be carried in the next chapter. With this particular expression, it is quite immediate to notice that each extra radiation carries a factor $\ln ^{2} N$, taking in account for soft and collinear emission, that implies, at $\mathrm{N}^{k} \mathrm{LO}$, LL terms of power $2 k$.

In literature, it is often common to distinguish in the expressions of $\bar{g}_{1}(\alpha)$, $\bar{g}_{2}(\alpha)$ different contributions according to their origin in diagrammatic calculation:

$$
\begin{align*}
& \int_{1}^{N^{2}} \frac{d n}{n} \int_{n \mu^{2}}^{Q^{2}} \frac{d q^{2}}{q^{2}} \bar{g}_{1}^{(i)}\left(\alpha_{s}\left(q^{2} / n\right)\right)+\int_{1}^{N} \frac{d n}{n} \int_{n \mu^{2}}^{Q p_{T}} \frac{d q^{2}}{q^{2}} \bar{g}_{2}^{(j)}\left(\alpha_{s}\left(q^{2} / n\right)\right)= \\
& \quad=2 \int_{0}^{1} d z \frac{z^{N-1}-1}{1-z}\left[D^{i}\left[Q^{2}(1-z)^{2}\right]+\int_{\mu^{2}}^{Q^{2}(1-z)^{2}} \frac{d q^{2}}{q^{2}} A^{i}\left[\alpha_{s}\left(q^{2}\right)\right]\right]  \tag{2.22}\\
& +\int_{0}^{1} d z \frac{z^{N-1}-1}{1-z}\left[B^{j}\left[Q p_{T}(1-z)^{2}\right]+\int_{\mu^{2}}^{Q p_{T}(1-z)} \frac{d q^{2}}{q^{2}} A^{j}\left[\alpha_{s}\left(q^{2}\right)\right]\right]
\end{align*}
$$

The anomalous dimension $A\left(\alpha_{s}\right)$ is often called cusp anomalous dimension and encloses all the most singular contributions $(1-z)_{+}^{-1}$ in the splitting functions as in Eq. (1.61). It coherently gives contributions both at the DY and DIS terms, unlike $B$ and $D$ that respectively collect hard collinear DIS contributions and soft but large-angle DY-like radiations ${ }^{2}$.

### 2.4. Phase space factorisation at small transverse momentum

In a way analogous to Sec. 2.2, in order to resum all the collinear logarithms we should make use of a phase space factorisation in the limit of small $p_{T}$. In $d=4-2 \epsilon$ dimensions, the phase space for the process Eq. (2.1) can be written as the product of phase spaces of each final parton times a delta factor of momentum conservation:

$$
\begin{align*}
d \Phi_{n+1}\left(p_{1}, p_{2} ; p_{H}, k_{1} \cdots, k_{n}\right)= & \frac{d^{3-2 \epsilon} p_{H}}{(2 \pi)^{3-2 \epsilon} 2 \sqrt{M^{2}+\left|p_{H}\right|^{2}}} \frac{d^{3-2 \epsilon} k_{1}}{(2 \pi)^{3-2 \epsilon} 2 E_{1}} \cdots \frac{d^{3-2 \epsilon} k_{n}}{(2 \pi)^{3-2 \epsilon} 2 E_{n}}  \tag{2.23}\\
& (2 \pi)^{4-2 \epsilon} \delta^{(4-2 \epsilon)}\left(p_{1}+p_{2}-p_{H}-k_{1 . .}-k_{n}\right)
\end{align*}
$$

We can then separate longitudinal and transverse momentum dependence through Sudakov parametrization

$$
\begin{equation*}
k_{i}=\left(1-z_{i}\right) \prod_{j=1}^{i-1} z_{j} \frac{p_{1}+p_{2}}{2}-\frac{k_{T_{i}}^{2} / s}{\left(1-z_{i}\right) \prod_{j=1}^{i-1} z_{j}} \frac{p_{1}-p_{2}}{2}+k_{T_{i}} \tag{2.24}
\end{equation*}
$$

[^6]to get the more explicit expression
\[

$$
\begin{align*}
& d \Phi_{n+1}\left(p_{1}, p_{2} ; p_{H}, k_{1} \ldots, k_{n}\right)=\frac{\left|p_{H}\right| d\left|p_{H}\right|\left(p_{T}^{2}\right)^{-\epsilon} d p_{T}^{2} d \Omega_{2-2 \epsilon}}{(2 \pi)^{3-2 \epsilon} 4 \sqrt{M^{2}+\left|p_{H}\right|^{2}} \sqrt{\left|p_{H}\right|^{2}-p_{T}^{2}}} \\
& \frac{d z_{1}\left(k_{T_{1}}^{2}\right)^{-\epsilon} d k_{T_{1}}^{2} d \Omega_{2-2 \epsilon}}{(2 \pi)^{3-2 \epsilon} 4 \sqrt{\left(1-z_{1}\right)^{2}-\frac{4}{\hat{s}} k_{T_{1}}^{2}}} \cdots \frac{d z_{n}\left(k_{T_{n}}^{2}\right)^{-\epsilon} d k_{T_{n}}^{2} d \Omega_{2-2 \epsilon}}{(2 \pi)^{3-2 \epsilon} 4 \sqrt{\left(1-z_{n}\right)^{2}-\frac{4}{\hat{s} z_{1}^{2} \ldots z_{n-1}^{2}} k_{T_{n}}^{2}}}  \tag{2.25}\\
& (2 \pi)^{4-2 \epsilon} \delta\left(\sqrt{\hat{s}}-\sqrt{M^{2}+\left|p_{H}\right|^{2}}-E_{1}-\ldots-E_{n}\right) \delta\left(p_{H_{z}}-k_{1_{z}}-\ldots-k_{n_{z}}\right)
\end{align*}
$$
\]

which shows a direct dependence on the transverse momenta $p_{T}, k_{T_{i}}$.
With this reparametrization, we recognize the splitting of the total momentum conservation condition into two deltas representing conservation of energy and longitudinal momentum. We already know from Sec. 2.2 that the first allows factorisation in Mellin space, so we must look for an analogous conjugate space for the second one. This is achieved by the exploiting the integral transforms of Sec. A.2, where angular integrations give the results:

$$
\begin{equation*}
\int d \Omega_{2-2 \epsilon} e^{i \vec{b} \cdot \overrightarrow{k_{T}}}=\left(b k_{T}\right)^{\epsilon}(2 \pi)^{1-\epsilon} J_{-\epsilon}\left(b k_{T}\right), \tag{2.26}
\end{equation*}
$$

We can then write a new factorizing expression for the phase space in the small $p_{T}$ limit $(b \rightarrow \infty)$ :

$$
\begin{align*}
d \Phi_{n+1}\left(p_{1}, p_{2} ; p_{H}, k_{1} \ldots, k_{n}\right)= & x \frac{\pi^{3-2 \epsilon}}{\Gamma(1-\epsilon)} d \xi_{p} \int d b^{2}\left(b p_{T}\right)^{-\epsilon} b^{-n \epsilon} J_{-\epsilon}\left(b p_{T}\right) \\
& J_{-\epsilon}\left(b k_{T_{1}}\right) \frac{M^{-\epsilon}\left(\xi_{1}\right)^{-\frac{\epsilon}{2}} d z_{1} d \xi_{1}}{4(2 \pi)^{2-\epsilon} \sqrt{\left.\left(1-z_{1}\right)^{2}-4 x \xi_{1}\right)}} \\
& \ldots  \tag{2.27}\\
& J_{-\epsilon}\left(b k_{T_{n}}\right) \frac{M^{-\epsilon}\left(\xi_{n}\right)^{-\frac{\epsilon}{2}} d z_{n} d \xi_{n}}{4(2 \pi)^{2-\epsilon} \sqrt{\left(1-z_{n}\right)^{2}-\frac{4 x}{z_{1}^{2} \ldots z_{n-1}^{2}} \xi_{n}}} \\
& \delta\left(x-z_{1} \ldots z_{n}\right)+O\left(\frac{1}{b}\right)
\end{align*}
$$

where we introduced the dimensionless variable $\xi_{i}=k_{T_{i}}^{2} / M^{2}$ such that $\xi_{i} \in\left[0 ; z_{1}^{2} \ldots z_{n i-1}^{2}(1-\right.$ $\left.\left.z_{i}\right)^{2} / 4 x\right]$.

In the small transverse momentum limit, all the $\xi_{i} \rightarrow 0$ and we can exploit the following distributional identity of Sec. A.3:

$$
\begin{equation*}
\lim _{\xi \rightarrow 0} \frac{1}{\sqrt{(1-z)^{2}-4 a \xi}}=\left[\frac{1}{1-z}\right]_{+}^{1}-\frac{1}{2} \ln \xi \delta(1-z) \tag{2.28}
\end{equation*}
$$

where we highlight an extra logarithmic contribution $(1-z)_{+}^{-1}$ coming directly from the factorised phase space. This identity allows for a decoupling of the variables $z_{i}, \xi_{i}$ and a final expression of the phase space:

$$
\begin{aligned}
d \Phi_{n+1}\left(p_{1}, p_{2} ; p_{H}, k_{1} \ldots, k_{n}\right)= & x \frac{\pi^{3-2 \epsilon}}{\Gamma(1-\epsilon)} d \xi_{p} \int d b^{2}\left(b p_{T}\right)^{-\epsilon} b^{-n \epsilon} J_{-\epsilon}\left(b p_{T}\right) \\
& J_{-\epsilon}\left(b k_{T_{1}}\right) \frac{M^{-\epsilon}\left(\xi_{1}\right)^{-\epsilon}}{(4 \pi)^{2}}\left[\left[\frac{1}{1-z_{1}}\right]_{+}^{z}-\frac{1}{2} \delta\left(1-z_{1}\right) \ln \xi_{1}\right] d z_{1} d \xi_{1}
\end{aligned}
$$

$$
\begin{align*}
& J_{-\epsilon}\left(b k_{T_{n}}\right) \frac{M^{-\epsilon}\left(\xi_{n}\right)^{-\frac{\epsilon}{2}}}{(4 \pi)^{2}}\left[\left[\frac{1}{1-z_{n}}\right]_{+}^{z}-\frac{1}{2} \delta\left(1-z_{n}\right) \ln \xi_{n}\right] d z_{n} d \xi_{n} \\
& \delta\left(x-z_{1} \ldots z_{n}\right)+O\left(\frac{1}{b}\right) \tag{2.29}
\end{align*}
$$

where the $\xi_{i} \mathrm{~S}$ range in $[0 ;+\infty]$ and transverse momentum distributions are factorised in FourierMellin space.

We should note that the jacobian factor

$$
\begin{equation*}
\frac{d E_{i}}{\sqrt{E_{i}^{2}-k_{T_{i}}^{2}}}=\frac{d z_{i}}{\sqrt{\left(1-z_{i}\right)^{2}-4 a \xi_{i}}} \tag{2.30}
\end{equation*}
$$

exhibits striking different behaviours in the two threshold $\left(z_{i} \rightarrow 1\right)$ and collinear $\left(\xi_{p} \rightarrow 0\right)$ limits, adding respectively an extra $\ln N$ and an extra $\ln b$ to the final resummed expression when confronted at fixed logarithmic order. This non-commutativity of the two limits constitutes the fundamental task of this thesis and will be deeply investigated in the next chapter.

### 2.5. Resummation at small transverse momentum

The results of the previous section show that, in order to perform effectively the resummation of collinear logarithms, we must do calculations in the Fourier-Mellin conjugate space. As reported in $[18-20]$, it will not surprise that, for the same arguments of Sec. 2.3, the coefficient function can be factorised in terms of:

$$
\begin{equation*}
C_{i j}\left(N, b, \alpha_{s}\left(M^{2}\right), M^{2}\right)=H_{i j}\left(N, \alpha_{s}\left(M^{2}\right)\right) S\left(N, b, M^{2}\right) \tag{2.31}
\end{equation*}
$$

where besides the hard function $H$ we have an exponential Sudakov form factor consequence of a rerun of the renormalization group argument of Eqs. (2.16) and following.

$$
\begin{equation*}
S\left(N, b, M^{2}\right)=\exp \left\{-\int_{\frac{1}{b^{2}}}^{M^{2}} \frac{d q^{2}}{q^{2}}\left[A^{p_{T}}\left(\alpha_{s}\left(q^{2}\right)\right) \ln \frac{M^{2}}{q^{2}}+B^{p_{T}}\left(\alpha_{s}\left(q^{2}\right), N\right)\right]\right\} \tag{2.32}
\end{equation*}
$$

One must recognize the resemblance of this exponent with the DIS term in Eq. (2.22), where $A^{p_{T}}$ still encloses the singular contributions of the splitting functions like the old cusp anomalous dimension $A\left(\alpha_{s}\right)$ did, but differs from it starting from NNLL due to the presence of an extra term $(1-z)_{+}^{-1}$ in the factorised phase space of Eq. (2.29). As in the case of soft resummation, the expansion of the $A$-terms at $\mathrm{N}^{k} \mathrm{LO}$ brings a LL factor of power $2 k$ due to the contribution of soft-collinear radiation.

If instead of the usual evaluation of the PDFs at the hard scale $Q^{2}$ we make the different choice of evaluating at the soft scale $\frac{1}{b^{2}}$, it is actually possible to get rid of the kinematics dependence in the Sudakov factor.

$$
\begin{equation*}
\bar{S}\left(b, M^{2}\right)=\exp \left\{-\int_{\frac{b_{0}^{2}}{b^{2}}}^{M^{2}} \frac{d q^{2}}{q^{2}}\left[\bar{A}^{p_{T}}\left(\alpha_{s}\left(q^{2}\right)\right) \ln \frac{M^{2}}{q^{2}}+\bar{B}^{p_{T}}\left(\alpha_{s}\left(q^{2}\right)\right)\right]\right\} \tag{2.33}
\end{equation*}
$$

This convenient resummation choice will be henceforth employed to display explicit resummed results coherently with refs. [17, 19, 20].

We can see that choice for the phase space in Eq. (2.29) correctly factorizes the coefficient function in Fourier-Mellin space and produces a sensible resummed expression. However, it does not reproduce the correct behaviour of the total cross section in the soft limit, as it was
noted in [19]. This fact can be easily verified with the explicit computation of the integral over $\xi_{1}$ (and in general over all the $\xi_{i} \mathrm{~s}$ ):

$$
\begin{align*}
\int_{0}^{\frac{(1-z)^{2}}{4}} d \xi \frac{1}{\sqrt{(1-z)^{2}-4 \xi}} & =\frac{1}{1-z} \int_{0}^{\frac{(1-z)^{2}}{4}} d \xi 1+\frac{2 \xi}{(1-z)^{2}}+\frac{6 \xi^{2}}{(1-z)^{4}}+\ldots  \tag{2.34}\\
& =\frac{(1-z)}{4}\left(1+\frac{1}{4}+\frac{1}{8}+\ldots\right)
\end{align*}
$$

All terms in the expansion in powers of $\xi_{i}$, when integrated, are of the same order and therefore not negligible as did in Eq. (2.29). A more relevant power expansion is then needed and some inspiration can be drawn from the inspection of the Fourier-Mellin transformed phase space:

$$
\begin{equation*}
\int_{0}^{1} d z z^{N-1} \int_{0}^{\frac{(1-z)^{2}}{4}} d \xi J_{0}(b M \sqrt{\xi}) \frac{1}{\sqrt{(1-z)^{2}-4 \xi}}=\frac{2}{b^{2} M^{2}}\left(1-\frac{N^{2}}{b^{2} M^{2}}+\frac{16 N^{4}}{b^{4} M^{4}}+\ldots\right) \tag{2.35}
\end{equation*}
$$

This expression suggests a better choice for the expansion parameter in terms of $N / b$. In this way, we can write another factorizing expression for the phase space that, unlike Eq. (2.29), does not spoil the soft limit when integrated:

$$
\begin{align*}
d \Phi_{n+1}\left(p_{1}, p_{2} ; p_{H}, k_{1} \ldots, k_{n}\right)= & x \frac{\pi^{3-2 \epsilon}}{\Gamma(1-\epsilon)} d \xi_{p} \int d b^{2}\left(b p_{T}\right)^{-\epsilon} b^{-n \epsilon} J_{-\epsilon}\left(b p_{T}\right) \\
& J_{-\epsilon}\left(b k_{T_{1}}\right) \frac{M^{-\epsilon}\left(\xi_{1}\right)^{-\frac{\epsilon}{2}}}{(4 \pi)^{2}} \frac{d z_{1} d \xi_{1}}{\sqrt{\left(1-z_{1}\right)^{2}-4 z_{1} \xi_{1}}} \\
& \ldots  \tag{2.36}\\
& J_{-\epsilon}\left(b k_{T_{n}}\right) \frac{M^{-\epsilon}\left(\xi_{n}\right)^{-\frac{\epsilon}{2}}}{(4 \pi)^{2}} \frac{d z_{n} d \xi_{n}}{\sqrt{\left(1-z_{n}\right)^{2}-4 z_{n} \xi_{n}}} \\
& \delta\left(x-z_{1} \ldots z_{n}\right)+O\left(\frac{1}{b}\right)+O\left(\frac{1}{N}\right)
\end{align*}
$$

where now $\xi_{i} \in\left[0 ;\left(1-z_{i}\right)^{2} / 4 z_{i}\right]$ and the kinematic limit considered is $b \rightarrow \infty$ at fixed $N / b$ rather than fixed $N$. The decoupling of the variables $z_{i}$ and $\xi_{i}$ will be pursued through the change of variables

$$
\begin{equation*}
z_{i}^{\prime}=z_{i}\left(\sqrt{1+\xi_{i}}+\sqrt{\xi_{i}}\right)^{2} \tag{2.37}
\end{equation*}
$$

while the singularities will be highlighted, as before, by means of Sec. A. 3 through the distributional identity

$$
\begin{gather*}
\frac{1}{\sqrt{(1-z)^{2}-4 z \xi}}=\frac{1}{\sqrt{\left(1-z^{\prime}\right)\left(1-(\sqrt{1+\xi}-\sqrt{\xi})^{4} z^{\prime}\right)}} \\
=\left[\frac{1}{\sqrt{\left(1-z^{\prime}\right)\left(1-(\sqrt{1+\xi}-\sqrt{\xi})^{4} z^{\prime}\right)}}\right]_{+}^{z}+\frac{1}{2(\sqrt{1+\xi}-\sqrt{\xi})^{2}}(\ln (1+\xi)-\ln \xi) \delta\left(1-z^{\prime}\right) . \tag{2.38}
\end{gather*}
$$

In this way, we get the modified expression for the phase space

$$
\int_{0}^{1} d x x^{N-1} \int d \xi_{1} d \xi_{2} \ldots d \xi_{n} \frac{d \Phi_{n+1}\left(p_{1}, p_{2} ; p_{H}, k_{1} \ldots, k_{n}\right)}{d \xi_{p}}=
$$

$$
\begin{align*}
& =\frac{\pi^{2-\epsilon} b^{n \epsilon}}{\Gamma(1-\epsilon)} \int d b^{2}\left(b p_{T}\right)^{-\epsilon} b^{-n \epsilon} J_{-\epsilon}\left(b p_{T}\right)\left(\sqrt{1+\xi_{p}}-\sqrt{\xi_{p}}\right)^{-2 N} \\
& \left\{\int_{0}^{\infty} d \xi\left(\sqrt{1+\xi_{p}}-\sqrt{\xi_{p}}\right)\right)^{2 N} J_{-\epsilon}\left(b k_{T}\right) M^{-\epsilon}(\xi)^{-\frac{\epsilon}{2}} \\
& \left.\int_{0}^{1} d z z^{N-1}\left[\frac{1}{\left.\sqrt{\left(1-z^{\prime}\right)\left(1-(\sqrt{1+\xi}-\sqrt{\xi})^{4} z^{\prime}\right.}\right)}\right]_{+}^{z}+\frac{\ln (1+\xi)-\ln \xi}{2(\sqrt{1+\xi}-\sqrt{\xi})^{2}} \delta\left(1-z^{\prime}\right)\right\}^{n} \\
& +O\left(\frac{1}{b}\right)+O\left(\frac{1}{N}\right) \tag{2.39}
\end{align*}
$$

To obtain this modified transverse momentum resummed transverse momentum distribution, we must remark that, unlike the phase space measure, matrix elements do not display further soft singularities in the collinear limit, hence they can be safely expanded in powers of $\xi_{p}$ at fixed $N$. For the modified analogous of Eq. (2.31) then, it will suffice to account for the use of the phase space of Eq. (2.39), with the result, in Mellin space:

$$
\begin{align*}
& C\left(N, \xi_{p}, \alpha_{s}\left(M^{2}\right), M^{2}\right)=\int_{0}^{\infty} d b \frac{b}{2} J_{0}\left(b M \sqrt{\xi_{p}}\right)\left(\sqrt{1+\xi_{p}}+\sqrt{\xi_{p}}\right)^{-2 N} \\
& \mathcal{H}_{i j}\left(N, \alpha_{s}\left(M^{2}\right)\right) \exp \left[d \xi(\sqrt{1+\xi}+\sqrt{\xi})^{2 N} J_{0}(b M \sqrt{\xi})\left[\frac{\mathcal{B}\left(N, \alpha_{s}\left(M^{2} \xi\right)\right)}{\xi}\right]_{+}^{p_{T}}+O\left(\frac{1}{b}\right)\right] \\
& \exp \left[\int_{0}^{\infty} d \xi(\sqrt{1+\xi}+\sqrt{\xi})^{2 N} J_{0}(b M \sqrt{\xi}) \int_{0}^{1} d z z^{N-1}\right. \\
& \left(\left[\frac{2 A^{p_{T}}\left(\alpha_{s}\left(M^{2} \xi\right)\right)}{\xi}\right]_{+}^{p_{T}}\left[\frac{1}{\sqrt{\left(1-z^{\prime}\right)\left(1-(\sqrt{1+\xi}-\sqrt{\xi})^{4} z^{\prime}\right)}}\right]_{+}^{z}+\delta(1-z) \frac{1}{2(\sqrt{1+\xi}-\sqrt{\xi})^{2}}\right. \\
& \left.\left.\left(2 A^{p_{T}}\left(\alpha_{s}\left(M^{2} \sqrt{\xi}\right)\right) \frac{\ln (1+\xi)}{\xi}-\left[\frac{2 A^{p_{T}}\left(\alpha_{s}\left(M^{2} \xi\right)\right) \ln \xi}{\xi}\right]_{+}^{p_{T}}\right)\right]+O\left(\frac{1}{b}\right)\right] \tag{2.40}
\end{align*}
$$

where $A^{p_{T}}$ is the same as in Eq. (2.31), while $\mathcal{B}$ and $\mathcal{H}_{i j}$ contains also contributions that are not enhanced in the limit $N \rightarrow \infty$.

## Matching Threshold and Transverse Momentum Resummations

From the results of the previous chapter, one may expect the threshold resummation of Eq. (2.20) and the transverse momentum resummation of Eq. (2.40) to be written in terms of the same $\ln b$ and $\ln N$, when considering the contemporary limit of soft and collinear emissions. In other words, we may expect the same expressions yield by the large $b$ limit of the transverse momentum resummed distribution that was obtained through the factorization at large $N$ and viceversa. This doesn't seem unlikely, as the additional added limit would only introduce relevant terms in the form of logarithms in the other variable. However, one may observe that the presence of terms in the form

$$
\begin{equation*}
\ln \left(N^{2}+b^{2}\right) \tag{3.1}
\end{equation*}
$$

introduces contributions that behave in a different way depending on the order in which the two limits were taken. These contributions actually rise from the integration of the jacobian Eq.(2.30), suggesting the possibility of the non-commutativity of the two limits. In the following sections, we will aim to establish a possible relationship between the two resummation techniques by performing explicit computations of the resummed transverse momentum distributions for the production of a Higgs boson through gluon fusion. This will be accomplished, in FourierMellin space, by matching fixed orders in powers of the strong coupling constant $\alpha_{s}$ of the two resummed expression.

### 3.1. The explicit threshold resummed result

In Eq. (2.21) we exposed a rather simple expression for the threshold resummed transverse momentum distribution particularly suitable for fixed order calculations:

$$
\begin{gather*}
C_{g g}\left(N, \frac{Q^{2}}{\mu^{2}}, \frac{Q p_{T}}{\mu^{2}}, \alpha_{s}\left(\mu^{2}\right)\right)=g_{0}^{g g}\left(\alpha_{s}\left(Q^{2}\right)\right) \exp \left[\frac{1}{\alpha_{s}\left(Q^{2}\right)} g_{1}^{g g}\left(\lambda, \frac{p_{T}}{Q}\right)\right. \\
\left.+g_{2}^{g g}\left(\lambda, \frac{p_{T}}{Q}\right)+\alpha_{s}\left(Q^{2}\right) g_{3}^{g g}\left(\lambda, \frac{p_{T}}{Q}\right)+O\left(\alpha_{s}^{3}\right)\right]  \tag{3.2}\\
=g_{0}^{g g}\left(\alpha_{s}\left(Q^{2}\right)\right)\left[1+\frac{1}{\alpha_{s}} g_{1}^{(2)}+g_{2}^{(1)}+\frac{1}{\alpha_{s}} g_{1}^{(3)}+g_{2}^{(2)}+\frac{1}{2 \alpha_{s}^{2}}\left(g_{1}^{(2)}\right)^{2}+\frac{1}{2}\left(g_{2}^{(1)}\right)^{2}+O\left(\alpha^{3}\right)\right] .
\end{gather*}
$$

This expression, with the definition of a convenient parameter $\lambda=\alpha_{s}\left(Q^{2}\right) \beta_{0} \ln N$ and $g_{i}=$ $\sum_{k} g_{i}^{(k)}$ such that $g_{i}^{(k)} \sim \alpha_{s}^{k}$, allows for immediate fixed order calculations starting from Eqs. (B.15-B.16).

For the calculations of the $g_{i}^{g g}(N, \xi)$ it sufficed to the perform the integrations of Eq. (2.22) where functions $A^{g}\left(\alpha_{s}\right), B^{g}\left(\alpha_{s}\right), D^{g}\left(\alpha_{s}\right)$ have been perturbatively expanded in terms of

$$
\begin{align*}
A^{g}\left(\alpha_{s}\right) & =\sum_{n=1}^{\infty} A_{n}\left(\frac{\alpha_{s}}{\pi}\right)^{n}  \tag{3.3}\\
B^{g}\left(\alpha_{s}\right) & =\sum_{n=1}^{\infty} B_{n}\left(\frac{\alpha_{s}}{\pi}\right)^{n}  \tag{3.4}\\
D^{g}\left(\alpha_{s}\right) & =\sum_{n=1}^{\infty} D_{n}\left(\frac{\alpha_{s}}{\pi}\right)^{n} \tag{3.5}
\end{align*}
$$

obtaining, up to NNLL, the results collected in Sec. B.2. Moreover, in the case of Higgs production, $g_{0}^{g g}\left(\alpha_{s}\left(Q^{2}\right)\right)$ is directly computed from diagrams and given, up to NLO accuracy, by Eq. (1.3.30) in [20]:

$$
\begin{align*}
\frac{d \sigma^{\mathrm{NLO}}}{d \xi}(x, \xi) & =\frac{\sigma_{0} \alpha_{s} C_{A}}{\pi}\left\{3 \zeta_{2} \delta(\xi) \delta(1-x)-\frac{4 x(1-x)^{2}-2 \xi x^{2}}{\sqrt{(1-x)^{2}-4 x \xi}}+2\left(1-x+x^{2}\right)^{2} D_{0}(x, \xi)\right\} \\
& =\frac{\sigma_{0} \alpha_{s} C_{A}}{\pi}\left\{3 \zeta_{2} \delta(\xi) \delta(1-x)+2 D_{0}(x, \xi)\right\}+O(\xi)+O(1-x) \\
\text { with } D_{0}(x, \xi) & =\left(\frac{1}{\sqrt{(1-x)^{2}-4 x \xi}}\right)_{+}^{z}\left(\frac{1}{\xi}\right)_{+}-\frac{1}{2} \delta(1-x)\left[\left(\frac{\ln \xi}{\xi}\right)_{+}-\frac{\ln (1+\xi)}{\xi}\right] . \tag{3.6}
\end{align*}
$$

We observe that, when expanded in powers of $\alpha_{s}$, the $g_{k}$ functions bring only extra $\ln \xi$ or $\ln N$ terms due to running coupling evolution, thus allowing to write the Fourier-Mellin transformed expression only in terms of transforms of some kind of distributions $D_{k}$ :

$$
\begin{equation*}
D_{k}(x, \xi)=\left(\frac{1}{\sqrt{(1-x)^{2}-4 x \xi}}\right)_{+}^{z}\left(\frac{\ln ^{k-1} \xi}{\xi}\right)_{+}-\frac{1}{2} \delta(1-x)\left[\left(\frac{\ln ^{k} \xi}{\xi}\right)_{+}-\frac{\ln (1+\xi)}{\xi}\right] \tag{3.7}
\end{equation*}
$$

These transforms, then, can be performed through the definition of a generating function

$$
\begin{align*}
\mathcal{G}_{1}(x, \xi, \epsilon)=(\xi)_{+}^{-1+\epsilon} & \left(\frac{1}{\sqrt{(1-x)^{2}-4 x \xi}}\right)_{+}^{z}-\frac{1}{2} \delta(1-x)\left[\left(\xi^{-1+\epsilon} \ln \xi\right)_{+}-\xi^{-1+\epsilon} \ln (1+\xi)\right] \\
\text { with } \quad G_{k, 1}(N, b)= & \int_{0}^{1} d x x^{N-1} \int_{0}^{\xi_{\max }} d \xi J_{0}(b Q \sqrt{\xi}) D_{k}(x, \xi) \\
& =\lim _{\epsilon \rightarrow 0} \frac{\partial^{k}}{\partial \epsilon^{k}} \int_{0}^{1} d x x^{N-1} \int_{0}^{\xi_{\max }} d \xi J_{0}(b Q \sqrt{\xi}) \mathcal{G}_{1}(x, \xi, \epsilon) \tag{3.8}
\end{align*}
$$

Explicit calculations of the $G_{k, 1}(N, b)$ can be extremely tricky and an exact result can only be provided in the case of $G_{0,1}$. Using the identities of the plus distribution:

$$
\begin{equation*}
\mathcal{G}_{1}(N, b, \epsilon)=\int_{0}^{1} d x x^{N-1} \int_{0}^{\xi_{\max }} d \xi J_{0}(b Q \sqrt{\xi}) \frac{\xi^{-1+\epsilon}}{\sqrt{(1-x)^{2}-4 x \xi}}-\frac{1}{2 \epsilon^{2}}-\frac{1}{\epsilon} \int_{0}^{1} d x x^{N-1}\left(\frac{1}{1-x}\right)_{+} \tag{3.9}
\end{equation*}
$$

whose integrals can be performed by Taylor expanding the Bessel function and integrating term by term. In this way, one gets

$$
\begin{array}{r}
\mathcal{G}_{1}(N, b, \epsilon)=\frac{1}{2} \sum_{p=0}^{\infty} \frac{\Gamma^{2}(p+\epsilon) \Gamma(N-p-\epsilon)}{\Gamma^{2}(p+1) \Gamma(N+p+\epsilon)}\left(-\frac{b^{2} M^{2}}{4}\right)^{p}-\frac{1}{2 \epsilon^{2}}+\frac{1}{\epsilon}(\digamma(N)+\gamma) \\
\Longrightarrow G_{0,1}(N, b)=\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p^{2}}\left(-\frac{b^{2} M^{2}}{4}\right)^{p} \frac{\Gamma(N-p)}{\Gamma(N+p)}+\frac{\pi^{2}}{12}+(\digamma(N)+\gamma)^{2} \tag{3.10}
\end{array}
$$

where $\digamma(N)$ is the usual digamma function, $\gamma$ the Euler constant and the limit was computed by means of [24]. At large $N$, it is possible to simplify this expression even more by exploiting the relation $\Gamma^{2}(p+\epsilon) / \Gamma^{2}(p+1)=1 / p^{2}$, obtaining:
$G_{0,1}(N, b) \sim \frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p^{2}}\left(-\frac{b^{2} M^{2}}{4 N^{2}}\right)^{p}+\frac{\pi^{2}}{12}+(\ln N+\gamma)^{2}=\frac{1}{2} \operatorname{Li}_{2}\left(-\frac{b^{2} M^{2}}{4 N^{2}}\right)+\frac{\pi^{2}}{12}+(\ln N+\gamma)^{2}$,
where we use the series definition of the dilogarithm. All the logarithmic divergences hidden in the dilogarithmm can then be made explicit by using identity (3.6) of ref. [25]:
$G_{0,1}(N, b)=(\ln N+\gamma)^{2}+\frac{1}{2} \operatorname{Li}_{2}\left(\frac{4 N^{2}}{4 N^{2}+b^{2} Q^{2}}\right)-\frac{1}{2} \ln \frac{b^{2} Q^{2}}{4 N^{2}} \ln \left(1+\frac{b^{2} Q^{2}}{4 N^{2}}\right)+\frac{1}{4} \ln ^{2}\left(1+\frac{b^{2} Q^{2}}{4 N^{2}}\right)$
which gives as a result an expression for transform with explicit dependencies on all the logarithms in $b$ and $N$.

### 3.2. The explicit transverse momentum resummed result

Explicit results in the collinear limit come directly form Eq. (2.40) as reported in [19, 20]. When adopting similar power expansions to Eq. (3.3), all the integrals in Eq. (2.40) can be expressed as combinations of these two types:

$$
\begin{align*}
G_{k, 1}(N, b)= & \int_{0}^{\infty} d \xi(\sqrt{1+\xi}+\sqrt{\xi})^{2 N} J_{0}(b M \sqrt{\xi}) \int_{0}^{1} d z z^{N-1} \\
& \left\{\left(\frac{\ln ^{k} \xi}{\xi}\right)_{+}\left[\frac{1}{\sqrt{\left(1-z^{\prime}\right)\left(1-(\sqrt{1+\xi}-\sqrt{\xi})^{4} z^{\prime}\right)}}\right]_{+}^{z}\right. \\
& \left.+\delta(1-z) \frac{1}{2(\sqrt{1+\xi}-\sqrt{\xi})^{2}}\left(\frac{\ln (1+\xi) \ln ^{k} \xi}{\xi}-\left(\frac{\ln ^{k+1} \xi}{\xi}\right)_{+}\right)\right\} \\
G_{k, 2}(N, b)= & \int_{0}^{\infty} d \xi(\sqrt{1+\xi}+\sqrt{\xi})^{2 N} J_{0}(b M \sqrt{\xi})\left(\frac{\ln ^{k} \xi}{\xi}\right)_{+} \tag{3.13}
\end{align*}
$$

As suggested by the notation, the $G_{k, 1} \mathrm{~s}$ are indeed the same of the previous section, with only inverted orders of integration. In a completely analogous way, we can then define two generating functions to perform the Fourier-Mellin transform

$$
\begin{equation*}
\mathcal{G}_{1}(N, b, \epsilon)=\frac{1}{2} \sum_{p=0}^{\infty} \frac{\Gamma^{2}(p+\epsilon) \Gamma(N-p-\epsilon)}{\Gamma^{2}(p+1) \Gamma(N+p+\epsilon)}\left(-\frac{b^{2} Q^{2}}{4}\right)^{p}-\frac{1}{2 \epsilon^{2}}+\frac{1}{\epsilon}(\digamma(N)+\gamma) \tag{3.14}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{G}_{2}(N, b, \epsilon)=\sum_{p=0}^{\infty} \frac{1}{2^{2 p+2 \epsilon-1}} \frac{\Gamma(N-p-\epsilon) \Gamma^{2}(p+\epsilon)}{\Gamma(N+1+p+\epsilon) \Gamma^{2}(p+1)}\left(-\frac{b^{2} Q^{2}}{4}\right)^{p} N-\frac{1}{\epsilon} . \tag{3.15}
\end{equation*}
$$

which in the large $b$ limit at fixed $\frac{N}{b}$ can be written in terms of hypergeometric functions exploiting the simplification of the coefficients depending on $\Gamma$

$$
\begin{align*}
& \mathcal{G}_{1}(N, b, \epsilon)=\frac{1}{2}\left(\frac{1}{N^{2}}\right)^{\epsilon} \Gamma^{2}(\epsilon){ }_{2} F_{1}\left(\epsilon, \epsilon, 1,-\frac{b^{2} M^{2}}{4 N^{2}}\right)-\frac{1}{2 \epsilon^{2}}+\frac{1}{\epsilon}(\ln (N)+\gamma)+O\left(\frac{1}{b}\right)  \tag{3.16}\\
& \mathcal{G}_{2}(N, b, \epsilon)=2^{1-2 \epsilon}\left(\frac{1}{N^{2}}\right)^{\epsilon} \Gamma(2 \epsilon)_{2} F_{1}\left(\epsilon, \frac{1}{2}+\epsilon, 1,-\frac{b^{2} M^{2}}{4 N^{2}}\right)-\frac{1}{\epsilon}+O\left(\frac{1}{b}\right) . \tag{3.17}
\end{align*}
$$

As stated before, calculations accurate at all logarithmic orders can not be carried outside of the case of $G_{0,1}$ exposed in the previous section and the analogous case of $G_{0,2}$ obtained exploiting [26]. In refs. [19, 20], however, very compact closed-form expressions are obtained by substituting the generating functions with suitable expressions which only differ by sub-leading terms. Since extra powers of $\ln \xi$ are obtained by differentiation with respect to $\epsilon$, an expression of the generating functions which reproduces transverse momentum resummation up to $\mathrm{NN}^{k} \mathrm{LL}$ can be obtained expanding the hypergeometric functions in powers of $\epsilon$, performing the limit $b \rightarrow \infty$ and then retaining the $k+1$ highest powers of $\ln b$. In the same way, because of the prefactor $N^{-2 \epsilon}$, an expression of the generating functions which reproduces the results in the large $N$ limit up to $\mathrm{N}^{j} \mathrm{LL}$ accuracy can be obtained by letting $b=0$ and then expanding the hypergeometric functions in powers of $\epsilon$ and retaining the first $j$ orders of the expansion.

Suitable interpolations of the hypergeometric functions fit for this purpose are then the following

$$
\begin{align*}
&{ }_{2} F_{1}(\epsilon, \epsilon, 1,-z)=\frac{(1+z)^{-\epsilon}}{\Gamma(1-\epsilon) \Gamma(\epsilon)}(\ln (1+z)-2 \gamma-\digamma(1-\epsilon)-\digamma(\epsilon)) \\
&+\epsilon^{2} \operatorname{Li}_{2}\left(\frac{1}{1+z}\right)+O(\mathrm{NNNLL})  \tag{3.18}\\
&{ }_{2} F_{1}\left(\epsilon, \frac{1}{2}+\epsilon, 1,-\frac{b^{2} M^{2}}{4 N^{2}}\right)=\frac{\sqrt{\pi} 2^{-2 \epsilon}(1+z)^{-\epsilon}}{\Gamma\left(\frac{1}{2}+\epsilon\right) \Gamma(1-\epsilon)}+O(\text { NNNLL }) \tag{3.19}
\end{align*}
$$

which imply accurate up to NNLL expressions of the $G_{k} \mathrm{~S}$ in terms of

$$
\begin{align*}
& G_{k, 1}(N, b)=\frac{(-1)^{k}}{2}\left[-\frac{\ln ^{k+2} \chi}{k+2}+\frac{\ln \bar{N}^{2}}{k+1} \ln ^{k+1} \chi+\ln ^{k} \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+O\left(\ln ^{j} \bar{N}^{2} \ln ^{k-j-1} \chi\right)\right]  \tag{3.20}\\
& G_{k, 2}(N, b)=-\frac{(-1)^{k}}{k+1} \ln ^{k+1} \chi+O\left(\ln ^{k-1} \chi\right) \tag{3.21}
\end{align*}
$$

with $\bar{N}=N e^{\gamma}, \chi=\bar{N}^{2}+\frac{\hat{b}^{2}}{b_{0}^{2}}, \hat{b}=b M, b_{0}=2 e^{-\gamma}$ and where we used the following equality from the appendix of [27].

$$
\begin{equation*}
(-1)^{k} \frac{\ln ^{k} \chi}{k}=-\int_{\frac{Q^{2}}{\chi}}^{Q^{2}} \frac{d q^{2}}{q^{2}} \ln ^{k-1}\left(\frac{q^{2}}{Q^{2}}\right) . \tag{3.22}
\end{equation*}
$$

The obtained $G_{k, 1}$ expressions then interpolate the large $N$ and large $b$ behaviour of Eq. (3.12) and the analogous exact expressions for $k>0$ and should work as adequate approximations in the two limits.

In a way analogous to Eq. (2.21), all these contributions can be resummed in $N, b$ space in terms of exponentials of some functions $g_{i}$, which are collected in Sec. B.3.

### 3.3. Matching the first fixed order

After having understood the structure of the resummed expression, we can now start with the matching procedure and try to answer the crucial question: are threshold resummation and transverse momentum resummation the same procedure?

We will start with computation at order $\alpha_{s}$ for the threshold resummed transverse momentum distribution and in all our analysis we will focus on the terms arising from the singularities, omitting logarithms originating from the scale dependence (ideally, we chose $\mu_{R}=\mu_{F}=M$ ). Looking at Eq. (3.2), first order results are only given by the NLO contributions of Eq. (3.6) not vanishing in the combined threshold and collinear limit. In the Fourier-Mellin space, this corresponds to the quantity:

$$
\begin{equation*}
T_{\mathrm{th}}^{\mathrm{I}}(N, b)=\frac{2 \alpha_{s} A_{1}}{\pi} G_{0,1}(N, b) . \tag{3.23}
\end{equation*}
$$

Explicit formulation in terms of logarithms of $N$ and $b$ can be carried out either by substituting in the interpolating expression of Eq. (3.20)

$$
\begin{aligned}
G_{0,1}(N, b)= & -\frac{1}{4} \ln ^{2} \chi+\frac{1}{2} \ln \bar{N}^{2} \ln \chi+\frac{1}{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
& \xrightarrow[N \rightarrow \infty]{\longrightarrow} \frac{1}{2} \zeta_{2}+\frac{1}{4} \ln ^{2} \bar{N}^{2} \\
& \longrightarrow b-\frac{1}{4} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}}+\frac{1}{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \bar{N}^{2},
\end{aligned}
$$

or the exact expression of Eq. (3.12)

$$
\begin{align*}
G_{0,1}(N, b) & =\frac{1}{4} \ln ^{2} \bar{N}^{2}+\frac{1}{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)-\frac{1}{2}\left(\ln \frac{\hat{b}^{2}}{b_{0}^{2}}-\ln \bar{N}^{2}\right)\left(\ln \chi-\ln \bar{N}^{2}\right)+\frac{1}{4}\left(\ln \chi-\ln \bar{N}^{2}\right)^{2} \\
& =\frac{1}{2} L^{2}\left(\frac{\bar{N}^{2}}{\chi}\right)-\frac{1}{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \chi+\frac{1}{2} \ln \frac{\hat{b}^{2}}{\bar{b}_{0}^{2}} \ln \bar{N}^{2}+\frac{1}{4} \ln ^{2} \chi \\
& \longrightarrow \underset{N \rightarrow \infty}{ } \frac{1}{2} \zeta_{2}+\frac{1}{4} \ln ^{2} \bar{N}^{2} \\
& \xrightarrow[b \rightarrow \infty]{\longrightarrow}-\frac{1}{4} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}}+\frac{1}{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \bar{N}^{2} . \tag{3.24}
\end{align*}
$$

We can observe that both expressions show the same asymptotic behaviour in large $N$ or $b$ limit, not only confirming the validity of the interpolation of ref. [19, 20], but also explicitly illustrating its subleading difference with respect to the exact result. The first order contribution, then, can be written in either of the following ways

$$
\begin{gather*}
T_{\text {th }}^{\mathrm{I}}(N, b)=-\frac{\alpha_{s} A_{1}}{2 \pi} \ln ^{2} \chi+\frac{\alpha_{s} A_{1}}{\pi} \ln \bar{N}^{2} \ln \chi+\frac{\alpha_{s} A_{1}}{\pi} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right),  \tag{3.25}\\
T_{\text {th }}^{\mathrm{I}(e x .)}(N, b)=\frac{\alpha_{s} A_{1}}{2 \pi} \ln ^{2} \chi-\frac{\alpha_{s} A_{1}}{\pi} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \chi+\frac{\alpha_{s} A_{1}}{\pi} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \bar{N}^{2}+\frac{\alpha_{s} A_{1}}{\pi} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)
\end{gather*}
$$

accordingly to the choice made, but always keeping the same asymptotic behaviour

$$
\begin{align*}
& \underset{N \rightarrow \infty}{\longrightarrow} \alpha_{s} \frac{A_{1}}{2 \pi} \ln ^{2} \bar{N}^{2}+\frac{\alpha_{s} A_{1}}{\pi} \zeta_{2} \\
& \underset{b \rightarrow \infty}{\longrightarrow}-\frac{\alpha_{s} A_{1}}{2 \pi} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}}+\frac{\alpha_{s} A_{1}}{\pi} \ln \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} . \tag{3.26}
\end{align*}
$$

The transverse momentum resummed result can be obtained simply through the interference of the hard function Eq. (B.19) with the first order expansion of the exponential in Eq. (2.22). Explicitly, $\alpha_{s}$ contributions will only come from the terms:

$$
\begin{equation*}
T_{p_{T}}^{\mathrm{I}}(N, b)=\frac{1}{\alpha_{s}} g_{1}^{(2)}(N, b)+g_{2}^{(1)}(N, b)+\frac{\alpha_{s} A_{1}}{\pi} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \tag{3.27}
\end{equation*}
$$

where with the notation $g_{i}^{(k)}$ we intend the terms of order $\alpha_{s}^{k}$ of the function $g_{i}$. Explicit values are then provided by the expansions of Eqs. (B.15-B.16)

$$
\begin{align*}
g_{1}^{(2)}(N, b) & =\alpha_{s}^{2} \frac{A_{1}}{\pi} \ln \bar{N}^{2} \ln \chi-\alpha_{s}^{2} \frac{A_{1}}{2 \pi} \ln ^{2} \chi  \tag{3.28}\\
g_{2}^{(1)}(N, b) & =0 \tag{3.29}
\end{align*}
$$

where coherently $g_{1}$ and $g_{2}$ do not show any term at order lower than 2 . We can observe that the thus obtained $T_{p_{T}}^{\mathrm{I}}(N, b)$ is exactly equal to one in Eq. (3.25).

This should not surprise since the exponent of App. B. 3 is defined by construction as the resummation of the interpolated expressions Eq. (3.20). This result, actually, is exactly what we expect to find when we expand the functions $A^{p_{T}}$ and $\mathcal{B}$ in Eq. (2.40) in $N, b$ space as in Eq. (3.3) and express the result in terms of $\alpha_{s}\left(M^{2}\right)$

$$
\begin{equation*}
\alpha_{s}\left(M^{2} \xi\right)=\frac{\alpha_{s}\left(M^{2}\right)}{1+\alpha_{s}\left(M^{2}\right) \beta_{0} \ln \xi}=-\alpha_{s}\left(M^{2}\right)+\alpha_{s}^{2}\left(M^{2}\right) \beta_{0} \ln \xi-\alpha_{s}^{3}\left(M^{2}\right) \beta_{0}^{2} \ln ^{2} \xi+\ldots \tag{3.30}
\end{equation*}
$$

by using running coupling evolution of Eq. (1.50). In this way, at precision $\alpha_{s}^{k}$ we have the expansion:

$$
\begin{align*}
A^{p_{T}}\left(\alpha_{s}\left(M^{2}\right), x, \xi\right)= & A_{1}\left(\frac{\alpha_{s}}{\pi}\right)+A_{2}\left(\frac{\alpha_{s}}{\pi}\right)^{2}-A_{1}\left(\frac{\alpha_{s}^{2} \beta_{0}}{\pi}\right) \ln \xi+A_{3}\left(\frac{\alpha_{s}}{\pi}\right)^{3} \\
& -A_{2}\left(\frac{\alpha_{s}^{3} \beta_{0}}{\pi^{2}}\right) \ln \xi+A_{1}\left(\frac{\alpha_{s}^{3} \beta_{0}^{2}}{\pi}\right) \ln ^{2} \xi+O\left(\alpha_{s}^{4}\right) \tag{3.31}
\end{align*}
$$

which contributes in the Sudakov exponent as

$$
\begin{align*}
& \exp \left[2 \sum_{i=0}^{\infty} A_{i}\left(\frac{\alpha_{s}}{\pi}\right)^{i}\left(\sum_{j=0}^{i-1} \alpha_{s}^{j} \beta_{0}^{j} G_{j, 1}(N, b)\right)\right]=1+\frac{2 \alpha_{s} A_{1}}{\pi} G_{0,1}(N, b) \\
& +\frac{2 \alpha_{s}^{2} A_{2}}{\pi^{2}} G_{0,1}(N, b)-\frac{2 \alpha_{s}^{2} A_{1} \beta_{0}}{\pi} G_{1,1}(N, b)+\frac{1}{2!}\left(\frac{2 \alpha_{s} A_{1}}{\pi} G_{0,1}(N, b)\right)^{2} \\
& +\frac{2 \alpha_{s}^{3} A_{3}}{\pi^{3}} G_{0,1}(N, b)-\frac{2 \alpha_{s}^{3} A_{2} \beta_{0}}{\pi^{2}} G_{1,1}(N, b)+\frac{2 \alpha_{s}^{3} A_{1} \beta_{0}^{2}}{\pi} G_{2,1}(N, b) \\
& +\frac{1}{2!}\left(\frac{8 \alpha_{s}^{3} A_{1} A_{2}}{\pi^{3}} G_{0,1}^{2}(N, b)\right)-\frac{1}{2!}\left(\frac{8 \alpha_{s}^{3} A_{1}^{2} \beta_{0}}{\pi^{2}} G_{0,1}(N, b) G_{1,1}(N, b)\right)+\frac{1}{3!}\left(\frac{2 \alpha_{s} A_{1}}{\pi} G_{0,1}(N, b)\right)^{3} \\
& +O\left(\alpha_{s}^{4}\right) \tag{3.32}
\end{align*}
$$

where the $\alpha_{s}$ term is correctly as in the form of Eq. (3.23).

### 3.4. Matching the second fixed order

First order sees perfect matching between the two different resummations when employed the transformed distribution $G_{k}$ of Eq. (3.20). Let us extend our analysis to the next order in order to confirm or overturn this result.

The $\alpha_{s}^{2}$ order of the threshold resummed transverse momentum distribution is given by the interference of the NLO $g_{0}^{g g}(N, \xi)$ of Eq. (3.6) with the first order expansion of the exponential Eqs. (B.15-B.16)

$$
\begin{equation*}
\frac{1}{\alpha_{s}} g_{1}^{(2)}(N, \xi)+g_{2}^{(1)}(N, \xi)=\alpha_{s} \frac{3 A_{1}}{8 \pi} \ln ^{2} \bar{N}^{2}-\alpha_{s} \frac{A_{1}}{2 \pi} \ln \bar{N}^{2} \ln \xi-\alpha_{s} \frac{B_{1}}{2 \pi} \ln \bar{N}^{2} . \tag{3.33}
\end{equation*}
$$

performing the transformation to conjugate space we get a contribution

$$
\begin{equation*}
T_{\mathrm{th}}^{\mathrm{II}}(N, b)=\alpha_{s}^{2} \frac{3 A_{1}^{2}}{4 \pi^{2}} \ln ^{2} \bar{N}^{2} G_{0,1}(N, b)-\alpha_{s}^{2} \frac{A_{1} B_{1}}{\pi^{2}} \ln \bar{N}^{2} G_{0,1}(N, b)-\alpha_{s}^{2} \frac{A_{1}^{2}}{\pi^{2}} \ln \bar{N}^{2} G_{1,1}(N, b), \tag{3.34}
\end{equation*}
$$

which assumes the following form when substituting in Eq. (3.20):

$$
\begin{align*}
T_{\mathrm{th}}^{\mathrm{II}}(N, b)= & -\alpha_{s}^{2} \frac{3 A_{1}^{2}}{16 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \chi+\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{3} \bar{N}^{2} \ln \chi+\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
& +\alpha_{s}^{2} \frac{A_{1} B_{1}}{4 \pi^{2}} \ln \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln \chi-\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \ln \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
& -\alpha_{s}^{2} \frac{A_{1}^{2}}{6 \pi^{2}} \ln \bar{N}^{2} \ln ^{3} \chi+\alpha_{s}^{2} \frac{A_{1}^{2}}{4 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \chi+\alpha_{s}^{2} \frac{A_{1}^{2}}{4 \pi^{2}} \ln ^{2} \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
\underset{N \rightarrow \infty}{\longrightarrow} & -\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \zeta_{2} \ln \bar{N}^{2}+\alpha_{s}^{2} \frac{5 A_{1}^{2}}{8 \pi^{2}} \zeta_{2} \ln ^{2} \bar{N}^{2}-\alpha_{s}^{2} \frac{A_{1} B_{1}}{4 \pi^{2}} \ln ^{3} \bar{N}^{2}+\alpha_{s}^{2} \frac{13 A_{1}^{2}}{48 \pi^{2}} \ln ^{4} \bar{N}^{2} \\
\underset{b \rightarrow \infty}{\longrightarrow} & -\alpha_{s}^{2} \frac{A_{1}^{2}}{6 \pi^{2}} \ln \bar{N}^{2} \ln ^{3} \frac{\hat{b}^{2}}{b_{0}^{2}}+\alpha_{s}^{2} \frac{A_{1}^{2}}{16 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}}+\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{3} \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \\
& +\alpha_{s}^{2} \frac{A_{1} B_{1}}{\ln ^{2}} \bar{N}^{2} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}}-\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \ln ^{2} \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}}} \tag{3.35}
\end{align*}
$$

This expression correctly reproduces the result of Eq. (B.9) in [17]. As in the previous section, the same asymptotic results are achieved when substituting Eq. (3.12), in spite of a different writing in terms of $\ln \chi$ :

$$
\begin{align*}
T_{\text {th }}^{\mathrm{II}(\text { ex. })}(N, b)= & \alpha_{s}^{2} \frac{3 A_{1}^{2}}{16 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \chi+\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \\
& +\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)-\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \ln \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \ln \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \chi \\
& -\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}}-\alpha_{s}^{2} \frac{A_{1} B_{1}}{4 \pi^{2}} \ln \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{2} \frac{A_{1}^{2}}{\pi} \ln \bar{N}^{2} G_{1,1}^{(\text {ex. })}(N, b) \tag{3.36}
\end{align*}
$$

where, of course, we do not know the exact expression for $G_{1,1}^{(\text {ex. })}(N, b)$.
The analogous result in transverse momentum resummation is given by the $\alpha_{s}^{2}$ term in the expansion of the exponential Eq. (B.20-B.21):

$$
T_{p_{T}}^{\mathrm{II}}(N, b)=\frac{1}{\alpha_{s}} g_{1}^{(3)}(N, b)+g_{2}^{(2)}(N, b)+\frac{1}{2}\left(\frac{1}{\alpha_{s}} g_{1}^{(2)}(N, b)\right)^{2}+\frac{\alpha_{s} A_{1}}{\pi} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) T_{p_{T}}^{\mathrm{I}}(N, b)
$$

with

$$
\begin{align*}
g_{1}^{(3)}(N, b) & =\alpha_{s}^{3} \frac{A_{1} \beta_{0}}{2 \pi} \ln \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{3} \frac{A_{1} \beta_{0}}{3 \pi} \ln ^{3} \chi  \tag{3.37}\\
g_{2}^{(2)}(N, b) & =\alpha_{s}^{2} \frac{A_{2}}{\pi^{2}} \ln \bar{N}^{2} \ln \chi-\alpha_{s}^{2} \frac{A_{2}}{2 \pi^{2}} \ln ^{2} \chi \tag{3.38}
\end{align*}
$$

that leads to the expression

$$
\begin{align*}
T_{p_{T}}^{\mathrm{II}}(N, b)= & \alpha_{s}^{2} \frac{A_{1} \beta_{0}}{2 \pi} \ln \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{2} \frac{A_{1} \beta_{0}}{3 \pi} \ln ^{3} \chi+\alpha_{s}^{2} \frac{A_{2}}{\pi^{2}} \ln \bar{N}^{2} \ln \chi-\alpha_{s}^{2} \frac{A_{2}}{2 \pi^{2}} \ln ^{2} \chi \\
& +\alpha_{s}^{2} \frac{A_{1}^{2}}{2 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{2} \frac{A_{1}^{2}}{2 \pi^{2}} \ln \bar{N}^{2} \ln ^{3} \chi+\alpha_{s}^{2} \frac{A_{1}^{2}}{8 \pi^{2}} \ln ^{4} \chi \\
& -\frac{\alpha_{s}^{2} A_{1}^{2}}{2 \pi^{2}} \ln ^{2} \chi \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \ln \bar{N}^{2} \ln \chi \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \mathrm{Li}_{2}^{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
\underset{N \rightarrow \infty}{\longrightarrow} & \alpha_{s}^{2} \frac{A_{2}}{2 \pi^{2}} \ln ^{2} \bar{N}^{2}+\alpha_{s}^{2} \frac{A_{1} \beta_{0}}{6 \pi} \ln ^{3} \bar{N}^{2}+\alpha_{s}^{2} \frac{A_{1}^{2}}{8 \pi^{2}} \ln ^{4} \bar{N}^{2}+\frac{\alpha_{s}^{2} A_{1}^{2}}{2 \pi^{2}} \zeta_{2} \ln ^{2} \bar{N}^{2}+\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \zeta_{2}^{2} \\
\underset{b \rightarrow \infty}{\longrightarrow} & \alpha_{s}^{2} \frac{A_{1} \beta_{0}}{2 \pi} \ln ^{2} \bar{N}^{2} \ln ^{2} \frac{\hat{b}^{2}}{\bar{b}_{0}^{2}}-\alpha_{s}^{2} \frac{A_{1} \beta_{0}}{3 \pi} \ln ^{3} \frac{\hat{b}^{2}}{b_{0}^{2}}+\alpha_{s}^{2} \frac{A_{2}}{\pi^{2}} \ln \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}}-\alpha_{s}^{2} \frac{A_{2}}{2 \pi^{2}} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}} \\
& +\alpha_{s}^{2} \frac{A_{1}^{2}}{2 \ln ^{2} \bar{N}^{2} \ln ^{2} \frac{\hat{b}^{2}}{b_{0}^{2}}-\alpha_{s}^{2} \frac{A_{1}^{2}}{2 \pi^{2}} \ln \bar{N}^{2} \ln ^{3} \frac{\hat{b}^{2}}{b_{0}^{2}}+\alpha_{s}^{2} \frac{A_{1}^{2}}{8 \pi^{2}} \ln ^{4} \frac{\hat{b}^{2}}{\bar{b}_{0}^{2}}} \text {. } \tag{3.39}
\end{align*}
$$

which is very different from the expression Eq. (3.35) in both limits. Indeed, despite exhibiting the same overall power of logarithms (all the $A_{1}^{2}$ terms have contributions $\sim \ln ^{4}$, while all the $A_{1} \beta_{0}$ have $\sim \ln ^{3}$ ), notable differences arise when considering the multiplicative constants and the distribution of the logarithms themselves between soft and collinear terms. Another significant difference is then the complete absence of the $A_{2}$ terms corresponding to double radiative corrections.

A part from this last difference, one may observe that, if the two limits do actually commute, as suggested by first order matching, this difference has only to rise from errors brought from the use of interpolated results Eq. (3.20) in place of exact expressions of the kind Eq. (3.12). If so, one may be tempted to derive the exact of expression of $G_{1,1}$ by simply comparing Eqs. (3.36-3.39). Under this assumption, we can obtain:

$$
\begin{align*}
\alpha_{s}^{2} \frac{A_{1}^{2}}{\pi} \ln \bar{N}^{2} G_{1,1}^{(\mathrm{ex.} .)}(N, b) \stackrel{?}{=} & \alpha_{s}^{2} \frac{5 A_{1}^{2}}{16 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \chi-\alpha_{s}^{2} \frac{A_{1}^{2}}{2 \pi^{2}} \ln \bar{N}^{2} \ln ^{3} \chi+\alpha_{s}^{2} \frac{A_{1}^{2}}{8 \pi^{2}} \ln ^{4} \chi \\
& +\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \ln \chi-\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln 3 \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \\
& -\frac{\alpha_{s}^{2} A_{1}^{2}}{2 \pi^{2}} \ln ^{2} \chi \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \ln \bar{N}^{2} \ln \chi \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
& -\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \operatorname{Li}_{2}^{2}\left(\frac{\bar{N}^{2}}{\chi}\right) \\
\xrightarrow[N \rightarrow \infty]{\longrightarrow} & -\alpha_{s}^{2} \frac{A_{1}^{2}}{16 \pi^{2}} \ln ^{4} \bar{N}^{2}+\alpha_{s}^{2} \frac{A_{1}^{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} \zeta_{2}+\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \zeta_{2}^{2} \\
\underset{b \rightarrow \infty}{\longrightarrow} & \alpha_{s}^{2} \frac{11 A_{1}^{2}}{16 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln 2 \frac{\hat{b}^{2}}{\bar{b}_{0}^{2}}-\alpha_{s}^{2} \frac{A_{1}^{2}}{2 \pi^{2}} \ln \bar{N}^{2} \ln ^{3} \frac{\hat{b}^{2}}{b_{0}^{2}}+\alpha_{s}^{2} \frac{A_{1}^{2}}{8 \pi^{2}} \ln ^{4} \frac{\hat{b}^{2}}{b_{0}^{2}} \\
& -\alpha_{s}^{2} \frac{3 A_{1}^{2}}{8 \pi^{2}} \ln ^{3} \bar{N}^{2} \ln \frac{\hat{b}^{2}}{b_{0}^{2}} \tag{3.40}
\end{align*}
$$

This expression, however, fails in respecting the asymptotic behaviour of $G_{1,1}$ given by Eq. (3.20) and thus cannot provide a reliable expression for the exact $G_{1,1}(N, b)$.

Recalling that $B_{1}=-\pi \beta_{0}$ and observing the difference displayed by $A_{1} B_{1}$ and $A_{2}$ terms, it seems that the two expressions actually derive from two different structures in terms of $G_{k, 1}$. Indeed, from Eq. (3.32), we can actually recognize the different structure of the transverse momentum resummed result:

$$
\begin{equation*}
T_{p_{T}}^{\mathrm{II}}(N, b)=\frac{2 \alpha_{s}^{2} A_{2}}{\pi^{2}} G_{0,1}(N, b)-\frac{2 \alpha_{s}^{2} A_{1} \beta_{0}}{\pi} G_{1,1}(N, b)+\frac{1}{2}\left(\frac{2 \alpha_{s} A_{1}}{\pi} G_{0,1}(N, b)\right)^{2} . \tag{3.41}
\end{equation*}
$$

### 3.5. Matching the third fixed order and beyond

Higher fixed order calculations for the threshold resummed expression can be obtained through the iteration of the scheme involved in the two previous sections. In fact, the $\alpha_{s}^{k}$-order expression can be derived through the interference between all the $\mathrm{N}^{j} \mathrm{LO} g_{0}^{g g}$ and the $i$-th expansion of the exponential given by Eqs. (B.15-B.17) such that $i+j=k$. In the case of $k=3$, the expansion of the exponential Eqs. (B.20-B.22) will give the relevant terms

$$
\begin{equation*}
\frac{1}{\alpha_{s}} g_{1}^{(3)}(N, b)+g_{2}^{(2)}+\frac{1}{2}\left(\frac{1}{\alpha_{s}} g_{1}^{(2)}+g_{2}^{(1)}\right)^{2}+\alpha_{s} g_{3}^{(1)} \tag{3.42}
\end{equation*}
$$

where, for simplicity we consider only contributions coming from $A$ and $B_{1}$ terms, obtaining:

$$
\begin{align*}
g_{1}^{(3)}(N, b)= & \alpha_{s}^{3} \frac{5 A_{1} \beta_{0}}{48 \pi} \ln ^{3} \bar{N}^{2}  \tag{3.43}\\
g_{2}^{(2)}(N, b)= & -\alpha_{s}^{2} \frac{A_{1} \beta_{0}}{16 \pi} \ln ^{2} \bar{N}^{2} \ln \xi+\alpha_{s}^{2} \frac{3 A_{2}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2}-\alpha_{s}^{2} \frac{B_{1} \beta_{0}}{8 \pi} \ln ^{2} \bar{N}^{2}  \tag{3.44}\\
\left(g_{1}^{(2)}(N, b)\right)^{2}= & \alpha_{s}^{4} \frac{9 A_{1}^{2}}{64 \pi^{2}} \ln ^{4} \bar{N}^{2}  \tag{3.45}\\
\left(g_{2}^{(1)}(N, b)\right)^{2}= & \alpha_{s}^{2} \frac{A_{1}^{2}}{4 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln ^{2} \xi+\alpha_{s}^{2} \frac{A_{1} B_{1}}{2 \pi^{2}} \ln ^{2} \bar{N}^{2} \ln \xi+\alpha_{s}^{2} \frac{B_{1}^{2}}{4 \pi^{2}} \ln ^{2} \bar{N}^{2}  \tag{3.46}\\
g_{1}^{(2)}(N, b) g_{2}^{(1)}(N, b)= & -\alpha_{s}^{3} \frac{3 A_{1}^{2}}{16 \pi^{2}} \ln ^{3} \bar{N}^{2} \ln \xi-\alpha_{s}^{2} \frac{3 A_{1} B_{1}}{16 \pi} \ln ^{2} \bar{N}^{2}  \tag{3.47}\\
g_{3}^{(1)}(N, b)= & -\alpha_{s} \frac{A_{1} \beta_{0}}{6} \pi \ln \bar{N}^{2}-\alpha_{s} \frac{A_{2}}{4 \pi^{2}} \ln \bar{N}^{2} \ln \xi \\
& +\alpha_{s} \frac{B_{1} \beta_{0}}{4 \pi} \ln \bar{N}^{2} \ln \xi-\alpha_{s} \frac{A_{1} \beta_{0}}{16 \pi} \ln \bar{N}^{2} \ln ^{2} \xi-\alpha_{s} \frac{A_{1} \beta_{0}}{24} \pi \ln \bar{N}^{2} . \tag{3.48}
\end{align*}
$$

The third order is then given by:

$$
\begin{aligned}
T_{\mathrm{th}}^{\mathrm{III}}(N, b)= & \alpha_{s}^{3} \frac{5 A_{1}^{2} \beta_{0}}{24 \pi^{2}} \ln ^{3} \bar{N}^{2} G_{0,1}(N, b)-\alpha_{s}^{3} \frac{A_{1}^{2} \beta_{0}}{8 \pi^{2}} \ln ^{2} \bar{N}^{2} G_{1,1}(N, b) \\
& +\alpha_{s}^{3} \frac{3 A_{1} A_{2}}{4 \pi^{3}} \ln ^{2} \bar{N}^{2} G_{0,1}(N, b)+\alpha_{s}^{3} \frac{9 A_{1}^{3}}{64 \pi^{3}} \ln ^{4} \bar{N}^{2} G_{0,1}(N, b) \\
& +\alpha_{s}^{3} \frac{A_{1}^{3}}{4 \pi^{3}} \ln ^{2} \bar{N}^{2} G_{2,1}(N, b)+\alpha_{s}^{3} \frac{A_{1}^{2} B_{1}}{2 \pi^{3}} \ln ^{2} \bar{N}^{2} G_{1,1}(N, b) \\
& +\alpha_{s}^{2} \frac{A_{1} B_{1}^{2}}{2 \pi^{3}} \ln ^{2} \bar{N}^{2} G_{0,1}(N, b)-\alpha_{s}^{3} \frac{3 A_{1}^{3}}{8 \pi^{3}} \ln ^{3} \bar{N}^{2} G_{1,1}(N, b) \\
& -\alpha_{s}^{3} \frac{3 A_{1}^{2} B_{1}}{8 \pi^{2}} \ln ^{3} \bar{N}^{2} G_{0,1}(N, b)-\alpha_{s}^{3} \frac{A_{1} A_{2}}{2 \pi^{3}} \ln \bar{N}^{2} G_{1,1}(N, b) \\
& +\alpha_{s}^{3} \frac{A_{1} B_{1} \beta_{0}}{2 \pi^{3}} \ln \bar{N}^{2} G_{1,1}(N, b)-\alpha_{s}^{3} \frac{A_{1}^{2} \beta_{0}}{8 \pi^{3}} \ln \bar{N}^{2} G_{2,1}(N, b)
\end{aligned}
$$

$$
\begin{equation*}
+\left(g_{0}^{\mathrm{NNLO}} \text { contributions }\right)+O(\mathrm{NLL}) \tag{3.49}
\end{equation*}
$$

As observed in the simpler NLO case, higher order expressions for $g_{0}^{g g}$ can be computed from diagrams whose number, however, increases drastically with the order. Regrettably, as of now, we still lack a suitable expression for $g_{0}^{\text {NNLO }}$ to perform a complete third-order matching.

The $k$-th fixed order calculations of the transverse momentum resummed transverse momentum distribution is analogously given by the interference between all the $i$-th order expansion of the exponential Eqs. (B. $20-$ B.22) and the $j$-th order of the hard function as reported in Eqs. (B.19) and following. In the case $k=3$, this implies the same derivation observed in Eq. (3.32):

$$
\begin{align*}
T_{p_{T}}^{\mathrm{III}}(N, b)= & \frac{2 \alpha_{s}^{3} A_{3}}{\pi^{3}} G_{0,1}(N, b)-\frac{2 \alpha_{s}^{3} A_{2} \beta_{0}}{\pi^{2}} G_{1,1}(N, b)+\frac{2 \alpha_{s}^{3} A_{1} \beta_{0}^{2}}{\pi} G_{2,1}(N, b) \\
& +\frac{4 \alpha_{s}^{3} A_{1} A_{2}}{\pi^{3}} G_{0,1}^{2}(N, b)-\frac{4 \alpha_{s}^{3} A_{1}^{2} \beta_{0}}{\pi^{2}} G_{0,1}(N, b) G_{1,1}(N, b)+\frac{4 \alpha_{s}^{3} A_{1}^{3}}{3 \pi^{3}} G_{0,1}^{3}(N, b) \\
& +\left(H_{g g}^{(1)}+\frac{\alpha_{s} A_{1}}{\pi} \mathrm{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)\right) T_{p_{T}}^{\mathrm{II}}(N, b) . \tag{3.50}
\end{align*}
$$

We observe that, coherently with the results of the previous section, $A_{3}$ and $A_{2} \beta_{0}$ terms do not appear in the threshold resummed expression, as if triple soft-collinear radiative corrections were, somehow, suppressed. As in the $\alpha_{s}^{2}$-order, the LL contributions continue exhibiting the same overall power of logarithms in both expressions ( $A_{1}^{3}: \sim \ln ^{6}, A_{1}^{2} \beta_{0}: \sim \ln ^{5}, A_{1} \beta_{0}^{2}: \sim \ln ^{4}$ and $A_{1} A_{2}: \sim \ln ^{4}$ ), while displaying different multiplicative constants and logarithmic distribution between soft and collinear contributions.

## Conclusions and outlook

In this work, our aim was to verify or disprove the possible commutativity between the limits of soft and collinear emissions in the case of Higgs production through gluon fusion. This was done by explicitly computing fixed orders in the coupling constant $\alpha_{s}$ of the inclusive transverse momentum distribution in the two different threshold or transverse momentum resummed expressions evaluated respectively at the large $b$ or $N$ limit. Through these calculations, we verified that these two limits do not commute.

Effects of the non-commutativity started to rise at second order, where, if we compare Eq. (3.34) and (3.41), we can notice an entire family of emergent terms in $A_{2}$ completely absent in the threshold resummed result. This can be seen as a statement of the fact that, somehow, double radiative corrections do not concur to the determination of the $\alpha_{s}^{2}$ order in the threshold limit. This may be the consequence of the phase space factorization of Fig. 2.1, where we imposed the existence of at least one non-soft parton recoiling on the produced Higgs. This implies that only single emissions can give rise to collinear singularities at order $\alpha_{s}^{2}$, since the other parton is forced to be non-collinear due to the recoil effect. Similar explanation can be given to suppression of the terms $A_{3}$ and $A_{2} \beta_{0}$ in Eq. (3.49).

Additionally, in the other terms, the dependence on the transformed distributions $G_{k, 1}$ seems like "downgraded", in the sense that $G_{k, 1}(N, b) \rightarrow \ln \bar{N}^{2} G_{k-1,1}(N, b)$. This phenomenon also appears to be a consequence of the different phase space factorization: it seems in fact that in each term we have somehow substituted a contribution originating a $\ln \chi$ (which can become both a soft or a collinear $\log$ ) with a $\ln N$ (which can only become soft, as the recoiling extra parton). This substitution however, doesn't seem to be a simple division/multiplication in $N, b$ space (otherwise we would have had $A_{2}$ terms also in the threshold resummed result), but something of the kind

$$
\begin{equation*}
\ln ^{j} N G_{k, 1} \xrightarrow{\mathfrak{S}} \frac{\ln ^{j+1} N}{j+1}(-1)^{k} k G_{k-1,1} \tag{3.51}
\end{equation*}
$$

which can be seen as an integration over the soft logarithm $A_{1} \ln N$ and a derivation over the collinear logarithm $A_{1} \ln \xi$, while the minus sign appears from the expansion in equation Eq. (3.30). The transformation thus built is able to explain the following results

$$
\begin{gather*}
\frac{2 \alpha_{s}^{2} A_{2}}{\pi^{2}} G_{0,1}(N, b) \xrightarrow{\mathfrak{S}} O\left(\frac{1}{b}\right)  \tag{3.52}\\
-\frac{2 \alpha_{s}^{2} A_{1} \beta_{0}}{\pi} G_{1,1}(N, b) \xrightarrow{\mathfrak{S}} \frac{2 \alpha_{s}^{2} A_{1} \beta_{0}}{\pi} \ln N G_{0,1}(N, b)=\frac{\alpha_{s}^{2} A_{1} \beta_{0}}{\pi} \ln \bar{N}^{2} G_{0,1}(N, b)
\end{gather*}
$$

However, it provides the following result for the $G_{0,1}^{2}$ term:

$$
\begin{gather*}
\frac{2 \alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} G_{0,1}^{2}(N, b)=\frac{2 \alpha_{s}^{2} A_{1}^{2}}{\pi^{2}}\left(-\frac{1}{2} G_{2,1}(N, b)-G_{1,1}(N, b) \ln \bar{N}^{2}+O\left(G_{0,1}(N, b)\right)\right) \\
\xrightarrow{\mathfrak{S}}-\frac{\alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} \ln \bar{N}^{2} G_{1,1}+\frac{\alpha_{s}^{2} A_{1}^{2}}{2 \pi^{2}} \ln ^{2} \bar{N}^{2} G_{0,1} . \tag{3.53}
\end{gather*}
$$

Analysing the difference in phase space factorization may provide some help: in the threshold resummed case we do not only assume the existence of one single parton recoiling, which
translates in a single application of the transformation $\mathfrak{S}$. Instead, we did not set any constraint on that number, so we may as well consider the contribution given by $\mathfrak{S}^{2}$

$$
\begin{equation*}
\frac{2 \alpha_{s}^{2} A_{1}^{2}}{\pi^{2}} G_{0,1}^{2}(N, b) \xrightarrow{\mathfrak{S}^{2}} \frac{\alpha_{s}^{2} A_{1}^{2}}{4 \pi^{2}} \ln ^{2} \bar{N}^{2} G_{0,1}, \tag{3.54}
\end{equation*}
$$

which, summed to the previous one, exactly provides the result in Eq. (3.34).
Since in the case of the terms $A_{2}$ and $A_{1} \beta_{0}$ all the higher $\mathfrak{S}^{k}$ contributions were identically zero, we can built a possible relation between the threshold and the transverse momentum resummed result in terms of

$$
\begin{equation*}
\overline{\mathfrak{S}}=\mathfrak{S}+\mathfrak{S}^{2}+\mathfrak{S}^{3}+\mathfrak{S}^{4}+\ldots \tag{3.55}
\end{equation*}
$$

which has to be verified by seeking for accordance in higher order calculations.
However, as stated at the end of the previous section, these calculations may reveal as extremely cumbersome due to the large amount of diagrammatic calculations required for the estimation of $g_{0}^{\mathrm{N}^{j} \mathrm{LO}}$. By assuming the validity of the transformation $\overline{\mathfrak{E}}$, one may derive the threshold resummed expression starting from the transverse momentum resummed one. By doing so, comparison between the obtained expression and the explicit results of the kind of Eq. (3.49) may provide possible ansätze for the $g_{0}^{\mathrm{N}^{j} \mathrm{LO}}$, later to be verified experimentally. Further studies, then can be conducted in order to determine the real transformation linking the two resummed expressions and obtain predictions of this kind.


## Integral Transforms and other useful mathematical objects

In this appendix, we will provide a brief overview of some mathematical objects encountered throughout the thesis. This collection will include key mathematical tools that have been utilized in the study, offering insights into their definitions, properties, and applications within the context of the research.

## A.1. Mellin transform

As we saw in Sec. 1.6, physical hadronic observables can always be written as multiplicative convolution of partonic luminosities and observables. As exploited in Eq. (1.57), these convolutions can actually factorize as products through a convenient choice of integral transforms. In our particular case, this is made possible by introduction of the Mellin transform:

$$
\begin{equation*}
f(N) \equiv \mathfrak{M}[f(z)] \equiv \int_{0}^{1} d z z^{N-1} f(z) \tag{A.1}
\end{equation*}
$$

First of all, we should note that this is nothing more than a special case of the Laplace transform, where we took $z=e^{-t}$.

$$
\begin{equation*}
f(s) \equiv \mathfrak{L}[f(t)] \equiv \int_{0}^{\infty} d t e^{-s t} f(t) \tag{A.2}
\end{equation*}
$$

which is convergent for $\Re s>c$, where c depends on $f(t)$ by the condition that at most $f(t) \sim e^{c t}$ int the limit $t \rightarrow \infty$. The inverse is then defined by the choice of some $c_{0}<c$ in the region of convergence as:

$$
\begin{equation*}
f(t)=\mathfrak{L}^{-1}[f(s)]=\frac{1}{2 \pi i} \int_{c_{0}-i \infty}^{c_{0}+i \infty} d s e^{-s t} f(s) \tag{A.3}
\end{equation*}
$$

which implies:

$$
\begin{equation*}
f(z)=\mathfrak{M}^{-1}[f(N)]=\frac{1}{2 \pi i} \int_{N_{0}-i \infty}^{N_{0}+i \infty} d N z^{-N} f(N) \tag{A.4}
\end{equation*}
$$

For a proper use of this mathematical tool, we will list some of the most important properties of this transform:

1. Shift operation

$$
\begin{equation*}
\mathfrak{M}\left[z^{c} f(z)\right]=f(N+c) \tag{A.5}
\end{equation*}
$$

2. Dilation, such as Hard Scale changing

$$
\begin{equation*}
\mathfrak{M}[f(a z) \Theta(a-x)]=\int_{0}^{a} d z z^{N-1} f(a z)=a^{N}=\int_{0}^{1} d \tau \tau^{N-1} f(\tau)=a^{N} f(N) \tag{A.6}
\end{equation*}
$$

with $a \in[0 ; 1]$
3. Logarithms of functions

$$
\begin{equation*}
\mathfrak{M}\left[\ln ^{k}(f(z))\right]=\frac{\partial^{k}}{\partial \epsilon^{k}} \mathfrak{M}\left[(f(z))^{\epsilon}\right] \equiv \frac{\partial^{k}}{\partial \epsilon^{k}} \mathcal{G}(N, \epsilon) \tag{A.7}
\end{equation*}
$$

where $\mathcal{G}$ is usually called generating function of $f(z)$.
4. Multiplicative convolutions

$$
\begin{align*}
\mathfrak{M}[(f \otimes g)(z)] & =\int_{0}^{1} d z z^{N-1} \int_{0}^{1} d y f(z) g\left(\frac{y}{z}\right) \\
& =\int_{0}^{1} d z z^{N-1} \int_{0}^{1} d y \int_{0}^{1} d w f(w) g(y) \delta(z-y w)  \tag{A.8}\\
& =\int_{0}^{1} d y y^{N-1} f(y) \int_{0}^{1} d w w^{N-1} g(w)=f(N) g(N)
\end{align*}
$$

## A.2. Multidimensional Fourier Transform

In Sec. 2.4, we needed some integral transform capable of factorizing addictive convolutions arising from integrals of the type:

$$
\begin{equation*}
\int d^{n} \xi \int d^{n} s f(\vec{\xi}) g(\vec{p}) \delta(\vec{k}-\vec{p}-\vec{\xi})=\int d^{n} \xi f(\vec{\xi}) g(\vec{\xi}-\vec{k}) \equiv(2 \pi)^{\frac{n}{2}}(f * g)(\vec{k}) \tag{A.9}
\end{equation*}
$$

In one-dimension, it is very well-known that such a factorization is effectively pursued by the Fourier transform and, in fact, if we introduce a general multidimensional Fourier tranform of a generic $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$

$$
\begin{equation*}
g(\vec{b}) \equiv \mathfrak{F}[g(\vec{\xi})] \equiv \frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} \xi e^{-i \vec{b} \cdot \vec{\xi}} g(\vec{\xi}), \tag{A.10}
\end{equation*}
$$

endowed of the inverse:

$$
\begin{equation*}
g(\vec{\xi}) \equiv \mathfrak{F}^{-1}[g(\vec{\xi})] \equiv \frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} b e^{i \vec{b} \cdot \vec{\xi}} g(\vec{\xi}), \tag{A.11}
\end{equation*}
$$

addictive convolutions factorize in any dimension $n$ :

$$
\begin{align*}
\mathfrak{F}[(f * g)(\vec{\xi})] & =\frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} \xi e^{-i \vec{b} \cdot \vec{\xi}} \frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} y f(\vec{y}) g(\vec{\xi}-\vec{y})=\overrightarrow{\vec{\xi}^{\prime}=\vec{\xi}-\vec{y}} \\
& =\frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} y e^{-i \vec{b} \cdot \vec{y}} f(\vec{y}) \frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} \xi^{\prime} e^{-i \vec{b} \cdot \vec{\xi}^{\prime}} g\left(\overrightarrow{\xi^{\prime}}\right)=f(\vec{b}) g(\vec{b}) . \tag{A.12}
\end{align*}
$$

In the widespread physical case of spherical symmetry, as in collider phenomenology, we may deal with central function $g(r)$ in $\mathbb{R}^{2}$, i.e. dependent only on the radius $r$, instead of general vector functions $g(\vec{\xi})$. In this case, Fourier transforms as in Eq. (A.10) are often called Bessel transform and significantly simplified in terms of:

$$
\begin{align*}
g(b)=\mathfrak{F}[g(r)] & =\frac{1}{(2 \pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n}} d^{n} \xi e^{-i \vec{b} \cdot \vec{\xi}} g(\vec{\xi})  \tag{A.13}\\
& =\int_{0}^{\infty} d r\left(\frac{r}{b}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(b r) f(r)
\end{align*}
$$

with $J_{\frac{n}{2}-1}$ being the Bessel function of order $\frac{n}{2}-1, b=|\vec{b}|$ and $r=|\vec{\xi}|$. Moreover, angular symmetry implies a simple writing for the inverse:

$$
\begin{equation*}
g(r)=\mathfrak{F}^{-1}[g(b)]=\int_{0}^{\infty} d b\left(\frac{b}{r}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(b r) f(b) \tag{A.14}
\end{equation*}
$$

The case of a transform over the 2 -dimensional space of tranverse momenta is particularly relevant in literature and is usually referred to as Hankel Transform:

$$
\begin{array}{r}
g(b) \equiv \mathfrak{H}[g(r)] \equiv \int_{0}^{\infty} r d r J_{0}(b r) g(r) \\
g(r) \equiv \mathfrak{H}^{-1}[g(b)] \equiv \int_{0}^{\infty} b d b J_{0}(b r) g(b) \tag{A.15}
\end{array}
$$

## A.3. Plus distribution

In Eq. (2.28) we first separate the jacobian deriving from phase factorisation in a divergent logarithmic term and a regular one exploiting the definition of the plus distribution.

This distribution is defined starting from a given function $f$ divergent in the limit $\xi \rightarrow 0$ by its action over a certain Schwartz test function $\phi(x)$

$$
\begin{equation*}
\int_{0}^{\xi_{\max }} d \xi[f(\xi)]_{+} \phi(\xi)=\int_{0}^{\xi_{\max }} d \xi f(\xi)[\phi(\xi)-\theta(1-\xi) \phi(0)] \tag{A.16}
\end{equation*}
$$

Another generalised plus distribution can be defined in the case of a function $g$ defined in the range $0<z<a$ and divergent in the limit $z \rightarrow a$

$$
\begin{equation*}
\int_{0}^{a} d z[g(z)]_{+}^{a} \phi(z)=\int_{0}^{a} d z g(z)[\phi(z)-\phi(a)] \tag{A.17}
\end{equation*}
$$

with the particular case $a=1$ often referred to as $[g(z)]_{+}^{z}$.
Formally, the two distributions behave as follows

$$
\begin{align*}
& {[f(\xi)]_{+}=\lim _{\eta \rightarrow 0^{+}}\left[\theta(\xi-\eta) f(\xi)-\delta(\xi) \int_{\eta}^{1} d \xi^{\prime} f\left(\xi^{\prime}\right)\right]}  \tag{A.18}\\
& {[g(z)]_{+}^{a}=\lim _{\eta \rightarrow 0^{+}}\left[\theta(a-z-\eta) f(z)-\delta(a-z) \int_{0}^{a-\eta} d z^{\prime} f\left(z^{\prime}\right)\right]}
\end{align*}
$$

where the limit is intended to be performed after the integration over the test function. These particular expressions in terms of delta functions show that plus distributions aim to regularize, i.e. make the integral finite, ordinary functions by "extracting" a delta where they diverge. Clearly, the plus distribution fails in regularizing all the divergent functions, but succeeds in the particularly relevant case of

$$
\begin{equation*}
\xi^{-\alpha} \quad(1-z)^{-\alpha} \quad \alpha<2 \tag{A.19}
\end{equation*}
$$

and in particular of all the powers of logarithms

$$
\begin{equation*}
\frac{\ln ^{k} \xi}{\xi} \quad \frac{\ln ^{k}(1-z)}{(1-z)} \tag{A.20}
\end{equation*}
$$

It is then possible to write general functions in terms of a regular and a divergent contribution as we did for the Eq. (2.28) and (2.38)

$$
\begin{align*}
& h(\xi)=[h(\xi)]_{+}+\delta(\xi) \int_{0}^{1} d \xi^{\prime} h\left(\xi^{\prime}\right) \\
& k(z)=[k(z)]_{+}^{a}+\delta(a-z) \int_{0}^{a} d z^{\prime} k\left(z^{\prime}\right) \tag{A.21}
\end{align*}
$$

With these definitions, it seems that plus distributions can't be more useful of other integral regularizing techniques such as the more famous Cauchy principal value. However, the bother of introducing a new, possibly cumbersome, regularization is richly rewarded by its simple expressions in conjugate space:

$$
\begin{align*}
& \mathfrak{H}\left[f(\xi)_{+}\right]=\frac{1}{2} \int_{0}^{\infty} d \xi\left[J_{0}(b \sqrt{\xi})-1\right] f(\xi) \\
& \mathfrak{M}\left[g(z)_{+}^{z}\right]=\int_{0}^{\infty} d z\left[z^{N-1}-1\right] g(z) \tag{A.22}
\end{align*}
$$

which allows all the calculations carried out in Chpt. 3.

## Explicit Resummed expressions

Within this dedicated appendix, the reader will find a collection of important coefficients and expressions that play a crucial role in performing soft and collinear resummations. This appendix offers a concise and focused resource, presenting the derived formulas and coefficients in the particular case of Higgs production through gluon fusion, where we omit contributions coming from scale dependence. General results can be found enlisted in refs. [17, 20].

## B.1. General coefficients

QCD Beta function:

$$
\begin{align*}
\mu^{2} \frac{d}{d \mu^{2}} \alpha_{s}\left(\mu^{2}\right) & =\beta\left(\alpha_{s}\right)=-\beta_{0} \alpha_{s}-\beta_{1} \alpha_{s}^{2}-\beta_{2} \alpha_{s}^{3}+O\left(\alpha_{s}^{4}\right)  \tag{B.1}\\
\beta_{0} & =\frac{11 C_{A}-4 T_{f} N_{f}}{2 \pi}  \tag{B.2}\\
\beta_{1} & =\frac{17 C_{A}^{2}-10 C_{A} N_{f} T_{f}-6 C_{F} N_{f} T_{f}}{24 \pi^{2}}  \tag{B.3}\\
\beta_{2} & =\frac{1}{128 \pi^{3}}\left(2857-\frac{5033}{9} N_{f}+\frac{325}{27} N_{f}^{2}\right) \tag{B.4}
\end{align*}
$$

Cusp Anomalous dimension $A$ [20, 28]:

$$
\begin{align*}
A_{1}^{g}= & C_{A}  \tag{B.5}\\
A_{2}^{g}= & \frac{C_{A}}{2}\left[\left(\frac{67}{18}-\zeta_{2}\right)-\frac{5}{9} N_{f}\right]  \tag{B.6}\\
\left(A_{3}^{g}\right)^{\text {th }}= & C_{A}\left[C_{A}^{2}\left(\frac{245}{96}-\frac{67}{36} \zeta_{2}+\frac{11}{24} \zeta_{3}+\frac{11}{20} \zeta_{2}^{2}\right)+C_{F} N_{f}\left(-\frac{55}{96}+2 \zeta_{3}\right)\right. \\
& \left.C_{A} N_{f}\left(-\frac{209}{432}+\frac{10}{36} \zeta_{2}-\frac{7}{12} \zeta_{2}\right)+N_{f}^{2}\left(-\frac{1}{108}\right)\right]  \tag{B.7}\\
\left(A_{3}^{g}\right)^{p_{T}}= & \frac{C_{A}}{4}\left[C_{A}^{2}\left(\frac{15503}{648}-\frac{67}{9} \zeta_{2}-11 \zeta_{3}+\frac{11}{2} \zeta_{4}\right)+C_{F} N_{f}\left(-\frac{55}{24}+2 \zeta_{3}\right)\right. \\
& \left.+C_{A} N_{f}\left(-\frac{2051}{324}+\frac{10}{9} \zeta_{2}\right)-\frac{25}{81} N_{f}^{2}\right] \tag{B.8}
\end{align*}
$$

DY-like coefficient $B$ [28]:

$$
\begin{equation*}
B_{1}^{g}=-\pi \beta_{0} \tag{B.9}
\end{equation*}
$$

$$
\begin{equation*}
B_{2}^{g}=\pi^{2} C_{A}^{2}\left[-\frac{611}{9}+\frac{88}{3} \zeta_{2}+16 \zeta_{3}\right]+C_{A} N_{f}\left[\frac{428}{27}-\frac{16}{3} \zeta_{2}\right]+2 C_{F} N_{f}-\frac{20}{27} N_{f}^{2} \tag{B.10}
\end{equation*}
$$

Hard function $H$ [20]:

$$
\begin{align*}
& H_{g g}^{(0)}=1  \tag{B.11}\\
& H_{g g}^{(1)}=\frac{3 \alpha_{s} C_{A} \zeta_{2}}{\pi}  \tag{B.12}\\
& H_{g g}^{(2)}=\left(\frac{\alpha_{s}}{\pi}\right)^{2}\left(C_{A}^{2}\left(\frac{93}{26}+\frac{67}{12} \zeta_{2}-\frac{55}{18} \zeta_{3}+\frac{65}{8} \zeta_{4}\right)+C_{A} N_{f}\left(-\frac{5}{3}-\frac{5}{6} \zeta_{2}-\frac{4}{9} \zeta_{3}\right)\right) \tag{B.13}
\end{align*}
$$

where $C_{A}=N_{c}$ and $C_{F}=\frac{N_{c}^{2}-1}{2 N_{c}}$ are the usual colour factors and $\zeta_{i}$ is the usual Riemann zeta function evaluated at integer values $i$.

## B.2. Explicit resummations at Threshold

The explicit resummed expression at Threshold for the Higgs production is given in Mellin space by the exponential:

$$
\begin{equation*}
g_{0}^{g g}(N, \xi) \exp \left[\sum_{i=1}^{\infty} \alpha_{s}^{i-2} g_{i}^{g g}(N, \xi)\right] \tag{B.14}
\end{equation*}
$$

with $g_{0}^{g g}(N, \xi)$ as in Eq. (3.6).
Explicit results for the $g_{i}^{g g}(N, \xi)$ are reported in [17] for the general case of fusion of partons of flavours $i$ and $j$ and here listed for the case of two gluons:

$$
\begin{align*}
g_{1}^{g g}\left(\lambda_{\bar{N}}, \frac{p_{T}}{Q}\right)= & \frac{A_{1}^{g}}{2 \beta_{0}^{2} \pi}\left[4 \lambda_{\bar{N}}+\left(1-2 \lambda_{\bar{N}}\right) \ln \left(1-2 \lambda_{\bar{N}}\right)+2\left(1-\lambda_{\bar{N}}\right) \ln \left(1-\lambda_{\bar{N}}\right)\right]  \tag{B.15}\\
g_{2}^{g g}\left(\lambda_{\bar{N}}, \frac{p_{T}}{Q}\right)= & \frac{A_{1}^{g}}{4 \beta_{0}^{3} \pi}\left[8 \lambda_{\bar{N}} \beta_{1}+\ln \left(1-2 \lambda_{\bar{N}}\right)\left(2 \beta_{1}+\beta_{1} \ln \left(1-2 \lambda_{\bar{N}}\right)\right)\right. \\
& \left.+4 \ln \left(1-\lambda_{\bar{N}}\right)\left(\beta_{1}+\beta_{0}^{2} \ln \frac{p_{T}}{Q}\right)+2 \beta_{1} \ln ^{2}\left(1-\lambda_{\bar{N}}\right)\right] \\
& -\frac{A_{2}^{g}}{2 \beta_{0}^{2} \pi^{2}}\left[4 \lambda_{\bar{N}}+\ln \left(1-2 \lambda_{\bar{N}}\right)+2 \ln \left(1-\lambda_{\bar{N}}\right)\right] \\
& +\frac{B_{1}^{g}}{\beta_{0} \pi} \ln \left(1-\lambda_{\bar{N}}\right)  \tag{B.16}\\
g_{3}^{g}\left(\lambda_{\bar{N}}, \frac{p_{T}}{Q}\right)= & \frac{1}{12 \beta_{0}^{4} \pi^{3}}\left\{\frac { 1 } { 2 \lambda _ { \overline { N } } - 1 } \left[6 \beta _ { 0 } \left(2 \beta_{0} \lambda_{\bar{N}}\left(-\lambda_{\bar{N}} A_{3}^{g}+\beta_{0} \pi D_{2}^{g}\right)\right.\right.\right. \\
& \left.+\pi A_{2}^{g}\left(2 \beta_{1} \lambda_{\bar{N}}\left(1+\lambda_{\bar{N}}\right) \lambda_{\bar{N}}+\beta_{1} \ln (1-2 \lambda)\right)\right) \\
& -2 \pi^{2} A_{1}^{g} \lambda_{\bar{N}}\left(6 \beta_{0} \beta_{2}\left(\lambda_{\bar{N}}-1\right)+6 \beta_{1}^{2} \lambda_{\bar{N}}+2 \pi^{2} \beta_{0}^{4}\right. \\
& \left.\left.-3 \ln \left(1-2 \lambda_{\bar{N}}\right)\left(2 \beta_{0} \beta_{2}+4 \beta_{1}^{2} \lambda_{\bar{N}}-4 \beta_{0} \beta_{2} \lambda_{\bar{N}}+\beta_{1}^{2} \ln \left(1-2 \lambda_{\bar{N}}\right)\right)\right)\right] \\
& +\frac{1}{\lambda_{\bar{N}}-1}\left[6 \beta _ { 0 } \left(\pi A_{2}^{g} \lambda_{\bar{N}}\left(\beta_{1}\left(2+\lambda_{\bar{N}}\right)+2 \beta_{0}^{2} \ln \frac{p_{T}}{Q}\right)\right.\right.
\end{align*}
$$

$$
\begin{align*}
& +\beta_{0} \lambda_{\bar{N}}\left(-\lambda_{\bar{N}} A_{3}^{g}+2 \pi\left(-\pi B_{1}^{g}\left(\beta_{1}+\beta_{0}^{2} \ln \frac{p_{T}}{Q}\right)+\beta_{0} B_{2}^{g}\right)\right) \\
& \left.+2 \beta_{1} \pi\left(A_{2}^{g}-\beta_{0} \pi B_{1}^{g}\right) \ln \left(1-\lambda_{\bar{N}}\right)\right) \\
& +\pi^{2} A_{1}^{g}\left(\lambda _ { \overline { N } } \left(6 \beta_{0} \beta_{2}\left(\lambda_{\bar{N}}-2\right)-6 \beta_{1}^{2} \lambda_{\bar{N}}-12 \beta_{0}^{2} \beta_{1} \ln \frac{p_{T}}{Q}\right.\right. \\
& \left.-\beta_{0}^{4}\left(-6 \ln ^{2} \frac{p_{T}}{Q}+\pi^{2}\right)\right)-6 \ln \left(1-\lambda_{\bar{N}}\right)\left(2 \left(\beta_{1}^{2} \lambda_{\bar{N}}+\beta_{0} \beta_{2}\left(1-\lambda_{\bar{N}}\right)\right.\right. \\
& \left.\left.\left.\left.\left.-\beta_{0}^{2} \beta_{1} \ln \frac{p_{T}}{Q}\right)+\beta_{1}^{2} \ln \left(1-\lambda_{\bar{N}}\right)\right)\right)\right]\right\} \tag{B.17}
\end{align*}
$$

with $\lambda_{\bar{N}}=\alpha_{s} \beta_{0} \ln \bar{N}$ and the coefficients given as above.

## B.3. Explicit resummations in the collinear limit

Explicit transverse momentum resummed results are listed in [20], however, in order to uniform notations, they are written hereafter using the same coefficients as in [28]. In Fourier-Mellin space, explicit results come in the form of Eqs. (2.31) and (2.33):

$$
\begin{equation*}
\bar{H}_{g g}(N, b) \exp \left[\sum_{i=1}^{\infty} \alpha_{s}^{i-2} g_{i}^{g g}(N, \xi)\right] \tag{B.18}
\end{equation*}
$$

where the hard function is written, up to NNLL, reabsorbing a dilogarithmic dependence from the Sudakov factor:

$$
\begin{equation*}
\bar{H}_{g g}(N, b)=H_{g g}(N, b)+\frac{\alpha_{s} A_{1}}{\pi} \operatorname{Li}_{2}\left(\frac{\bar{N}^{2}}{\chi}\right)+O(\text { NNNLL }) \tag{B.19}
\end{equation*}
$$

The $g_{i}(N, b)$ are then:

$$
\begin{align*}
g_{1}^{g g}\left(\lambda_{\bar{N}^{2}}, \lambda_{\chi}\right)= & \frac{A_{1}^{g}}{\pi \beta_{0}^{2}}\left(\lambda_{\chi}+\left(1-\lambda_{\bar{N}^{2}}\right) \ln \left(1-\lambda_{\chi}\right)\right)  \tag{B.20}\\
g_{2}^{g g}\left(\lambda_{\bar{N}^{2}}, \lambda_{\chi}\right)= & \frac{A_{1}^{g} \beta_{1}}{\pi \beta_{0}^{3}}\left[\left(1-\lambda_{\bar{N}^{2}}\right) \frac{\lambda_{\chi}+\ln \left(1-\lambda_{\chi}\right)}{1-\lambda_{\chi}}+\frac{1}{2} \ln \left(1-\lambda_{\chi}\right)^{2}\right] \\
& -\frac{A_{2}^{g}}{\pi^{2} \beta_{0}^{2}} \frac{\left(1-\lambda_{\bar{N}^{2}}\right) \lambda_{\chi}+\left(1-\lambda_{\chi}\right) \ln \left(1-\lambda_{\chi}\right)}{1-\lambda_{\chi}}  \tag{B.21}\\
g_{3}^{g g}\left(\lambda_{\bar{N}^{2}}, \lambda_{\chi}\right)= & \frac{A_{1}^{g} \beta_{1}^{2}}{2 \pi \beta_{0}^{4}}\left[\frac{\lambda_{\chi}+\ln \left(1-\lambda_{\chi}\right)}{\left(1-\lambda_{\chi}\right)^{2}}\left(\lambda_{\chi}+\left(1-2 \lambda_{\chi}\right) \ln \left(1-\lambda_{\chi}\right)\right)\right] \\
& +\frac{A_{1}^{g} \beta_{2}}{\pi \beta_{0}^{3}}\left[\frac{\left(2-3 \lambda_{\chi}\right) \lambda_{\chi}}{2\left(1-\lambda_{\chi}\right)^{2}}+\ln \left(1-\lambda_{\chi}\right)\right] \\
& -\frac{A_{2}^{g} \beta_{1}}{\pi^{2} \beta_{0}^{3}}\left[\frac{\left(2-3 \lambda_{\chi}\right) \lambda_{\chi}}{2\left(1-\lambda_{\chi}\right)^{2}}+\frac{\left(1-2 \lambda_{\chi}\right) \ln \left(1-\lambda_{\chi}\right)}{\left(1-\lambda_{\chi}\right)^{2}}\right] \\
& -\frac{A_{3}^{g}}{2 \pi^{3} \beta_{0}^{2}} \frac{\lambda_{\chi}^{2}}{\left(1-\lambda_{\chi}\right)^{2}}-\frac{B_{2}^{g}}{\pi^{2} \beta_{0}} \frac{\lambda_{\chi}}{1-\lambda_{\chi}} \\
& +\frac{A_{1}^{g}}{\pi} \operatorname{Li}_{2} \frac{\bar{N}^{2}}{\chi} \frac{\lambda_{\bar{N}^{2}}}{1-\lambda_{\bar{N}^{2}}}-\frac{A_{1}^{g} \beta_{1}^{2}}{\pi \beta_{0}^{4}} \frac{\lambda_{\bar{N}^{2}}\left(\lambda_{\chi}^{2}+\ln ^{2}\left(1-\lambda_{\chi}\right)\right)}{4\left(1-\lambda_{\chi}\right)^{2}}
\end{align*}
$$

$$
\begin{align*}
& \frac{A_{1}^{g} \beta_{2}}{\pi \beta_{0}^{3}} \frac{\lambda_{\bar{N}^{2}} \lambda_{\chi}}{2\left(1-\lambda_{\chi}\right)^{2}}-\frac{A_{2}^{g} \beta_{1}}{\pi^{2} \beta_{0}^{3}} \frac{\lambda_{\bar{N}^{2}}\left(\lambda_{\chi}\left(2-\lambda_{\chi}\right)+2 \ln \left(1-\lambda_{\chi}\right)\right)}{2\left(1-\lambda_{\chi}\right)} \\
& +\frac{A_{3}^{g}}{\pi^{3} \beta_{0}^{2}} \frac{\lambda_{\bar{N}^{2}}\left(2-\lambda_{\chi}\right) \lambda_{\chi}}{2\left(1-\lambda_{\chi}\right)^{2}}-\frac{D_{2}^{g}}{4 \beta_{0}} \lambda_{\bar{N}^{2}} \\
& \frac{A_{1}^{g} \beta_{1}}{\pi \beta_{0}^{2}} \frac{\lambda_{\bar{N}^{2}}\left(2 \lambda_{\chi}\left(\lambda_{\chi}-2\right)+\ln \left(1-\lambda_{\chi}\right)\right)}{\left(1-\lambda_{\chi}\right)^{2}} \tag{B.22}
\end{align*}
$$

with

$$
\begin{equation*}
\lambda_{\bar{N}^{2}}=\alpha_{s} \beta_{0} \ln \bar{N}^{2} \quad \text { and } \quad \lambda_{\chi}=\alpha_{s} \beta_{0} \ln \chi=\alpha_{s} \beta_{0} \ln \left(\bar{N}^{2}+\frac{\hat{b}^{2}}{b_{0}^{2}}\right) \tag{B.23}
\end{equation*}
$$

## References

[1] S. Weinberg. The Quantum Theory of Fields - Volume I. Cambridge University Press, (1995).
[2] S. Weinberg. The Quantum Theory of Fields - Volume II. Cambridge University Press, (1995).
[3] M. E. Peskin and D. V. Schroeder. An Introduction to Quantum Field Theory. AddisonWesley, (1995).
[4] T. Lancaster and S. J. Blundell. Quantum Field Theory for the Gifted Amateur. Oxford University Press, (2014).
[5] G. Parisi and Y. Wu. Perturbative theory without gauge fixing. Scientia Sinica 24, 483 (1981).
[6] L. D. Faddeev and V. N. Popov. Phys. Lett. 25B, 29. Oxford University Press, (1967).
[7] A. Rogers. Supermanifolds - Theory and Applications. World Scientific, (2007).
[8] J. D. Bjorken and E. A. Paschos. Inelastic Electron-Proton and $\gamma$-Proton Scattering and the Structure of the Nucleon. Phys. Rev. 185, (1969) 1975.
[9] R. P. Feynman. Very High-Energy Collisions of Hadrons. Phys. Rev. Lett. 23, (1969) 1415.
[10] Y. L. Dokshitzer. Sov. Phys. JETP 46, (1977) 641 [Zh. Eksp. Teor. Fiz. 73 (1977) 1216].
[11] V. N. Gribov and L. N. Lipatov. Sov. J. Nucl. Phys. 15, (1972) 438 [Yad. Fiz. 15 (1972) 781].
[12] G. Altarelli and G. Parisi. Asimptotic Freedom in Parton Language. Nucl. Phys. B 126, (1977) 298.
[13] T. Kinoshita. Mass singularities of Feynman amplitudes. J.Math.Phys. 3. (1962).
[14] T. Lee and M. Nauenberg. Degenerate Systems and Mass Singularities. Phis. Rev. 133. (1964).
[15] M. Grazzini A. Ilnicka M. Spira M. Wiesemann. Modeling BSM effects on the Higgs transverse- momentum spectrum in an EFT approach. JHEP 03, (2017).
[16] D. de Florian A. Kulesza and W. Vogelsang. Threshold resummation for high-transversemomentum Higgs production at the LHC. JHEP 0602, (2006) 047.
[17] S. Forte G. Ridolfi and S. Rota. Threshold resummation of transverse momentum distributions beyond next-to-leading log. J. High Energ. Phys. 2021, 110 (2021).
[18] J. C. Collins D. E. Soper and G. F. Sterman. Transverse Momentum Distribution in Drell-Yan Pair and W and Z Boson Production. Nucl. Phys. B 250, (1985) 199.
[19] C. Muselli S. Forte and G. Ridolfi. Combined threshold and transverse momentum resummation for inclusive observables. J. High Energ. Phys. 2017, 106 (2017).
[20] C. Muselli. Resummations of Transverse Momentum Distributions. PhD Thesis, (2017).
[21] S. Forte and G. Ridolfi. Renormalization group approach to soft gluon resummation. Nucl. Phys. B 650, (2003) 229.
[22] C. Bollini and J. J. Giambiagi. Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter. Il Nuovo Cimento B 12 (1), (1972).
[23] G. 't Hooft and M. Veltman. Regularization and renormalization of gauge fields. Nucl. Phys. B 44 (1), (1972).
[24] T. Huber and D. Maitre. HypExp: A Mathematica package for expanding hypergeometric functions around integer-valued parameters. Comput. Phys. Commun. 175, (2006) 122.
[25] L. C. Maximon. The dilogarithm function for complex argument. The Royal Society, (2003).
[26] T. Huber and D. Maitre. HypExp 2, Expanding Hypergeometric Functions about HalfInteger Parameters. Comput. Phys. Commun. 178, (2008) 755.
[27] M. Bonvini S. Forte and G. Ridolfi. Borel resummation of transverse momentum distributions. Nucl. Phys. B 808, (2009) 347.
[28] S. Moch J.A.M. Vermaseren and A. Vogt. Higher-order corrections in threshold resummation. Nuclear Physics B 726, (2005) 317-315.


[^0]:    ${ }^{1}$ These matrices are elements of the Lie group of linear transformations of the 4 -dimensional real vector space that leave invariant the Minkowski metric (of signature ( 1,3 )). In order to represent isometries accurately, we need to focus on a specific subgroup of these transformations, namely the subgroup with a determinant of 1 . This particular subgroup is referred to as the indefinite special orthogonal group $S O(1,3)$.

[^1]:    ${ }^{2}\left[a_{i}, a_{j}\right]_{\mp}=\left[a_{i}^{\dagger}, a_{j}^{\dagger}\right]_{\mp}=0,\left[a_{i}, a_{j}^{\dagger}\right]_{\mp}=\delta_{i j}$, where the sign depends on weather we are considering fermions or bosons

[^2]:    ${ }^{3}$ Indeed, various versions of the Dirac equation exist based on different choices of the representation. In this case, we used the Weyl or chiral representation. The name "chiral", actually, is particularly meaningful about the underlying structure of this representation which, actually, is not truly irreducible. From the block structure of the generators Eq. (1.22), we may be tempted to write the Dirac field as:

    $$
    \psi=\binom{\psi_{R}}{\psi_{L}}
    $$

    In this setting, general fermions are written in terms of these two two-components spinors that, akin to hands, can't be sent one into another by the action of Lorentz transformation. Therefore, they are referred to as right-handed and left-handed Weyl spinors, respectively, as they possess fixed helicity. This distinction in helicity plays a significant role in characterizing the behavior of fermions in the context of particle physics.

[^3]:    ${ }^{4}$ Grassmann or anticommuting numbers $\eta_{i}$ are often exploited in QFT to easily describe fermionic fields due to their properties:

    $$
    \eta_{i} \eta_{j}=-\eta_{j} \eta_{i} \quad x_{i} \eta_{j}=\eta_{j} x_{i} \quad\left(\eta_{i}\right)^{2}=0
    $$

[^4]:    ${ }^{5}$ By performing Mellin transform, product convolutions turn into simple products $\mathcal{L}(x) \otimes \hat{\sigma}(x) \rightarrow A_{N} C_{N}$, see Sec. A. 1

[^5]:    ${ }^{1}$ It is often convenient, in the determination of phase space measures $d \phi$, to adopt dimensional regularisation techniques $[22,23]$ where the dimension of phase space is made vary through a parameter $\epsilon$ allowing for the convergence of otherwise divergent integrals and analytically continue the found solution in the limit $\epsilon \rightarrow 0$.

[^6]:    ${ }^{2}$ starting at $O\left(\alpha_{s}^{2}\right)$, thus contributing from NNLO

